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Decomposition with a Mixed Model When the Trend-Cycle Component Is Quadratic

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Abstract

This study develops a rigorous procedure for the decomposition of a time series whose underlying structure follows a mixed model with a quadratic trend-cycle component. Conventional approaches, namely subtracting the sum of the estimated trend and seasonal components from the observed series or dividing the observed series by their product, often fail to produce an adequate decomposition under mixed model assumptions. To address this limitation, a method based on the statistical properties of the Buys–Ballot table is proposed. When the appropriate trend-cycle component is quadratic, estimates of the trend parameters and seasonal indices are derived from the row and column means and variances of the Buys–Ballot table. The results show that the proposed approach provides an adequate fit to the observed data and yields stable and interpretable seasonal indices. The method is therefore recommended for the decomposition of time series exhibiting quadratic trend-cycle behaviour under mixed models.

1 Introduction

The purpose of descriptive time series analysis is the decomposition of an observed series into its component parts (that is, to isolate the components from an observed series). Theoretically, an observed time series $(X_t, t = 1, 2, \ldots, n)$ is conceived as consisting of four components representing the trend (T_t) (long-term direction), the seasonal (S_t) (systematic, calendar-related movements), cyclical (C_t) (long-term oscillations or swings about the trend), and irregular (e_t) (unsystematic, short-term fluctuations) components (Kendall and Ord [1]; Chatfield [2]). The three models most commonly used for decomposition are as follows:
Additive Model:

$$X_t = T_t + S_t + C_t + e_t.$$

Multiplicative Model:

$$X_t = T_t \times S_t \times C_t \times e_t.$$

Mixed Model (Pseudo-Additive):

$$X_t = T_t \times S_t \times C_t + e_t.$$

If short periods of time are involved, the cyclical component is often absorbed into the trend (Chatfield [2]), and the observed time series $(X_t, t = 1, 2, \ldots, n)$ can be decomposed into the trend-cycle component

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(M_t), seasonal component (S_t), and the irregular component (e_t). Therefore, the decomposition models are restated as follows:
Additive Model:

$$X_t = M_t + S_t + e_t. \tag{1}$$

Multiplicative Model:

$$X_t = M_t \times S_t \times e_t. \tag{2}$$

Mixed Model (Pseudo-Additive):

$$X_t = M_t \times S_t + e_t. \tag{3}$$

It is always assumed that the seasonal effect, when it exists, has period s, that is, it repeats after s time periods:

$$S_{t+s} = S_t, \quad \text{for all } t.$$

For (1), it is convenient to assume that the sum of the seasonal components over a complete period is zero:

$$\sum_{j=1}^{s} S_{t+j} = 0.$$

Similarly, for (2) and (3), the corresponding assumption is that the sum of the seasonal components over a complete period is s:

$$\sum_{j=1}^{s} S_{t+j} = s.$$

It is also assumed that the irregular component e_t is Gaussian white noise. Specifically, $e_t \sim N(0, \sigma_1^2)$ for (1) and (3), while for (2), $e_t \sim N(1, \sigma_2^2)$. For (1)–(3), it is assumed that $\mathrm{Cov}(e_t, e_{t+k}) = 0$ for all $k \neq 0$.

The irregular component e_t is assumed to be independently and identically distributed with zero mean and constant variance, i.e., $e_t \sim N(0, \sigma^2)$. This assumption ensures unbiased and efficient estimation of the quadratic trend parameters and permits exact finite-sample inference. The structural results derived in subsequent sections require only the weaker condition of zero mean and finite variance; normality is invoked primarily for inferential convenience.

The traditional practice in descriptive time series analysis is to obtain estimates of the components present in the observed series and isolate them as much as possible. The first step is usually to estimate and eliminate M_t for each time period from the observed data, either by subtraction for (1) or by division for (2). The de-trended series is obtained as $X_t - \hat{M}_t$ for (1), or $X_t / \hat{M}_t$ for (2) and (3). The seasonal effect is then obtained by estimating the average of the de-trended series at each season.

The de-trended, de-seasonalized series is obtained as $X_t - \hat{M}_t - \hat{S}_t$ for (1), or $X_t / (\hat{M}_t \hat{S}_t)$ for (2) and (3). Thus, the estimate of the residual (irregular component) is

$$\hat{e}_t = X_t - \hat{M}_t - \hat{S}_t,$$

for the additive model, and

$$\hat{e}_t = \frac{X_t}{\hat{M}_t \hat{S}_t},$$

for the multiplicative model.

Having fitted a model to a time series, one often examines whether the model adequately describes the observed pattern. This is typically achieved by analyzing the residuals for properties of a purely random process (residual analysis). For details, see Box et al. [3] and Linde [4].

From the foregoing, the residual estimate in the mixed model cannot be derived by subtracting or dividing the joint estimates of the trend-cycle and seasonal components from the observed series. Consequently, when a series admits the mixed model, the conventional decomposition steps may not be applicable.

For the mixed model (3), the de-trended and de-seasonalized series obtained by subtraction is

$$\hat{e}_t = X_t - \hat{M}_t - \hat{S}_t = M_t S_t + e_t - (\hat{M}_t + \hat{S}_t),$$

and by division

$$\hat{e}_t = \frac{X_t}{\hat{M}_t \hat{S}_t} = \frac{M_t S_t}{\hat{M}_t \hat{S}_t} + \frac{e_t}{\hat{M}_t \hat{S}_t}.$$

Thus, neither subtraction nor division yields the desired residual. This may explain why many analysts avoid the mixed model, even when it provides the best fit to the data.

Previous studies on time series decomposition have largely focused on additive and multiplicative models. The additive model assumes linear separability of components, while the multiplicative model assumes proportional relationships among components. The Buys–Ballot table has been widely used as a descriptive tool for estimating seasonal effects and identifying trend patterns. Several researchers have extended its application to parameter estimation in time series models. However, limited attention has been given to decomposition under mixed models, particularly when the trend-cycle component is nonlinear. This study contributes to the literature by addressing this gap through a quadratic trend specification within a mixed model framework.

Some related literature, according to Iwueze et al. [5], indicates that the additive model performs well when the amplitude of both irregular and seasonal variations remains constant as the trend level changes. When the amplitude of both seasonal and irregular variations increases with the trend level, the multiplicative model is more appropriate. Only additive and multiplicative models were examined in that study; the mixed model was not considered.

Iwueze and Nwogu [6] showed that the column variance for an additive model depends only on the trend parameter, whereas for a multiplicative model, it depends on both the trend parameter and seasonal indices. Only one test was developed for the additive model. Model selection between additive and multiplicative forms may be achieved by comparing seasonal and column variances, as well as using row, column, and overall means and variances from the Buys–Ballot table. Chatfield [2] also recommended the use of the Buys–Ballot table for detecting trend and seasonal effects in time series data. Nwogu et al. [7] considered mixed models only in cases where the trend is linear.

Therefore, the objective of this study is to develop a decomposition procedure for time series that admit the mixed model. The Buys–Ballot approach is used to derive estimates of the trend parameters and seasonal indices from the row, column, and overall means and variances of the observed series. The remainder of the study is organized as follows: Section 2 presents the theoretical development, where Section 2.1 provides the expressions for the row, column, and overall means and variances, Section 2.2 derives the estimates of the trend parameters and seasonal indices, and Section 2.3 outlines the proposed decomposition procedure. Section 3 illustrates the application of the proposed method using simulated data, while Sections 4 and 5 present the discussion and conclusion, respectively.

2 Methodology

2.1 Theoretical Framework

The Buys–Ballot [8] table is a two-dimensional arrangement of an observed series with m rows and s columns, corresponding to the number of years/periods and the length of the periodic interval, together with totals and averages. This procedure has been developed for the choice of model for time series decomposition, the estimation of trend parameters and seasonal indices, and the selection of appropriate transformations based on the row, column, and overall means and variances. The method adopted in this study is the Buys–Ballot procedure proposed by Iwueze et al. [6] and Iwueze and Nwogu [9].

A seasonal time series of length n is arranged into m rows and s columns. When the series is arranged in this way, as shown in Table 1, the rows represent the years/periods, while each column represents a season within a period/year (e.g., months in a year). A cell (i, j) of the table contains the observed value corresponding to period i and season j. To analyze the data, the row and column totals ($T_{i.}$ and $T_{.j}$), row

and column means ($\bar{X}_{i.}$ and $\bar{X}_{.j}$), row and column standard deviations ($\hat{\sigma}_{i.}$ and $\hat{\sigma}_{.j}$), and the overall mean and standard deviation ($\bar{X}_{..}$ and $\hat{\sigma}_{..}$) are used; see Iwueze et al. [5].

Table 1: The Buys–Ballot table.

Row (i)	**Season (j)**								
	1	2	$\cdots$	j	$\cdots$	s	$T_{i.}$	$\bar{X}_{i.}$	$\hat{\sigma}_{i.}$
1	X_1	X_2	$\cdots$	X_j	$\cdots$	X_s	$T_{1.}$	$\bar{X}_{1.}$	$\hat{\sigma}_{1.}$
2	X_{s+1}	X_{s+2}	$\cdots$	X_{s+j}	$\cdots$	X_{2s}	$T_{2.}$	$\bar{X}_{2.}$	$\hat{\sigma}_{2.}$
3	X_{2s+1}	X_{2s+2}	$\cdots$	X_{2s+j}	$\cdots$	X_{3s}	$T_{3.}$	$\bar{X}_{3.}$	$\hat{\sigma}_{3.}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$X_{(i-1)s+1}$	$X_{(i-1)s+2}$	$\cdots$	$X_{(i-1)s+j}$	$\cdots$	X_{is}	$T_{i.}$	$\bar{X}_{i.}$	$\hat{\sigma}_{i.}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$X_{(m-1)s+1}$	$X_{(m-1)s+2}$	$\cdots$	$X_{(m-1)s+j}$	$\cdots$	X_{ms}	$T_{m.}$	$\bar{X}_{m.}$	$\hat{\sigma}_{m.}$
$T_{.j}$	T_1	T_2	$\cdots$	$T_{.j}$	$\cdots$	$T_{.s}$	$T_{..}$	–	–
$\bar{X}_{.j}$	$\bar{X}_1$	$\bar{X}_2$	$\cdots$	$\bar{X}_{.j}$	$\cdots$	$\bar{X}_{.s}$	–	$\bar{X}_{..}$	–
$\hat{\sigma}_{.j}$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\cdots$	$\hat{\sigma}_{.j}$	$\cdots$	$\hat{\sigma}_{.s}$	–	–	$\hat{\sigma}_{..}$

The notation used in Table 1 is defined as follows:
$T_{i.}$ = Total for the ith period/year.
$T_{.j}$ = Total for the jth season.
$T_{..}$ = Sum of all observations.
$\bar{X}_{i.}$ = Mean of the ith period/year.
$\bar{X}_{.j}$ = Mean of the jth season.
$\bar{X}_{..}$ = Overall mean of the observations.
$\hat{\sigma}_{i.}$ = Standard deviation for the ith period/year.
$\hat{\sigma}_{.j}$ = Standard deviation for the jth season.
$\hat{\sigma}_{..}$ = Overall standard deviation of the observations.
s = Number of seasons per period/year.
m = Number of periods/years.
$n = ms$ = Number of observations.

2.2 Estimation Procedure

The trend-cycle component is assumed to follow a quadratic form:

$$T_t = a + bt + ct^2,$$

where a, b, and c are parameters to be estimated. In terms of the Buys–Ballot table, equation (3) may be rewritten as

$$X_{(i-1)s+j} = \left\{a + b\left[(i-1)s + j\right] + c\left[(i-1)s + j\right]^2\right\} S_{(i-1)s+j} + e_{(i-1)s+j},$$

with $t = (i-1)s + j$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, s$. Following Iwueze and Nwogu [9] and Iwueze and Ohakwe [10], the row, column, and overall means and variances shown in Table 2 are obtained. The estimates of the trend parameters and seasonal indices are then derived from these row, column, and overall means and variances of the Buys–Ballot table.

Table 2: Row, column and overall means and variances for the mixed model with quadratic trend-cycle component (see Nzenwa [11]).

Measure	Estimate
$\bar{X}_{i.}$	$a + bC_1 + cC_2 + s[b + 2cC_1](i-1) + cs^2(i-1)^2 + \bar{e}_{i.}$
$\bar{X}_{.j}$	$\left\{a + \frac{(n-s)b}{2} + \frac{(n-s)(2n-s)c}{6} + [b + c(n-s)]j + cj^2\right\} S_j + \bar{e}_{.j}$
$\bar{X}_{..}$	$a + \frac{(n-s)b}{2} + \frac{(n-s)(2n-s)c}{6} + [b + c(n-s)]C_1 + cC_2 + \bar{e}_{..}$
$\sigma^2_{i.}$	$\sigma^2 + \Big[a^2\sigma^2_{S_j} + b^2\sigma^2_{jS_j} + c^2\sigma^2_{j^2S_j} + 2ab\,\mathrm{Cov}(S_j, jS_j) + 2ac\,\mathrm{Cov}(S_j, j^2S_j)$ $+ 2bc\,\mathrm{Cov}(jS_j, j^2S_j)\Big] + 2s\Big[ab\sigma^2_{S_j} + 2bc\sigma^2_{jS_j} + (b + 2ac)\mathrm{Cov}(S_j, jS_j)$ $+ bc\,\mathrm{Cov}(S_j, j^2S_j) + 2c^2\mathrm{Cov}(jS_j, j^2S_j)\Big](i-1) + s^2\Big[(b^2 + 2ac)\sigma^2_{S_j}$ $+ 4c^2\sigma^2_{jS_j} + 6bc\,\mathrm{Cov}(S_j, jS_j) + 2c^2 s\,\mathrm{Cov}(jS_j, j^2S_j)\Big](i-1)^2$ $+ \Big[s^2(2bcs\sigma^2_{S_j} + 4c^2\mathrm{Cov}(S_j, jS_j))\Big](i-1)^3 + (c^2s^4\sigma^2_{S_j})(i-1)^4$
$\sigma^2_{.j}$	$\sigma^2 + B_jS^2_j$
$\frac{n-1}{n}\sigma^2_x$	$\frac{n^2-s^2}{180}\left\{15(b + 2cC_1)^2 + 30(n-s)(b + 2cC_1)c + (2n-s)(8n-11s)c^2\right\}$ $+ \left\{\left[a + \frac{(n-s)b}{2} + \frac{(n-s)(2n-s)c}{6}\right]^2 + \frac{n^2-s^2}{12}[b + c(n-s)c]^2\right.$ $\left. + \frac{n^2-s^2}{180}[n^2 - (2s)^2]c^2\right\}\sigma^2_{S_j}\left\{[b + (n-s)c]^2 + \frac{(n-s)^2c^2}{3}\right\}\sigma^2_{jS_j}$ $+ c^2\sigma^2_{j^2S_j} + 2\Big[ab + \frac{(n-s)}{2}b^2 + (n-s)ac + \frac{(n-s)(2n-s)}{2}bc$ $+ \frac{n(n-s)^2}{2}c^2\Big]\mathrm{Cov}(S_j, jS_j) + 2\Big[ac + \frac{(n-s)}{2}bc + \frac{(n-s)(2n-s)}{6}c^2\Big]$ $\times\,\mathrm{Cov}(S_j, j^2S_j) + +2\Big[bc + (n-s)c^2\Big]\mathrm{Cov}(jS_j, j^2S_j) + \sigma^2$

The quantity B_j appearing in Table 2 is given by

$$B_j = \frac{n(n+s)}{180}\left\{15b^2 + 30(n-s)bc + (2n-s)(8n-11s)c^2 + 60[bc + (n-s)c^2]j + 60c^2j^2\right\}.$$

2.3 Estimates of Trend Parameters and Seasonal Indices

From Table 2, the row mean is given by

$$\begin{aligned}
\bar{X}_{i.} &= \frac{1}{s}\sum_{j=1}^{s}[a + bj + cj^2]S_j + \frac{1}{s}\sum_{j=1}^{s}[b + 2cj]S_j(i-1) \\
&\quad + \frac{1}{s}cs^2\sum_{j=1}^{s}S_j(i-1)^2 + \bar{e}_{i.} \\
&= \frac{1}{s}\sum_{j=1}^{s}\left\{[a + bj + cj^2] + [b + 2cj](i-1) + cs^2(i-1)^2\right\}S_j + \bar{e}_{i.} \\
&= a + bC_1 + cC_2 + s[b + 2cC_1](i-1) + cs^2(i-1)^2 + \bar{e}_{i.}, \qquad (4)
\end{aligned}$$

where

$$C_1 = \frac{1}{s}\sum_{j=1}^{s}jS_j,$$

and

$$C_2 = \frac{1}{s}\sum_{j=1}^{s}j^2S_j.$$

The row mean in (4) is a quadratic function of $(i-1)$ and can be written as

$$\bar{X}_{i.} = \alpha + \beta(i-1) + \gamma(i-1)^2 + \bar{e}_{i.},$$

where

$$\alpha = a + bC_1 + cC_2,$$
$$\beta = s[b + 2cC_1], \tag{5}$$

and

$$\gamma = cs^2. \tag{6}$$

From (6), we have

$$\hat{c} = \frac{\gamma}{s^2}.$$

Substituting into (5) gives

$$\hat{b} = \frac{\beta}{s} - 2\hat{c}C_1. \tag{7}$$

In the absence of estimates of the seasonal effects, a trial estimate of b is obtained by substituting $S_j = 1$ into (7) as

$$\hat{b} = \frac{\beta}{s} - \frac{2\hat{c}}{s}\sum_{j=1}^{s} jS_j$$
$$\approx \frac{\beta}{s} - \frac{2\hat{c}}{s}\sum_{j=1}^{s} j = \frac{\beta}{s} - \frac{2\hat{c}}{s}\cdot\frac{s(s+1)}{2}.$$

Thus, a trial estimate of b is

$$\hat{b} \approx \frac{\beta}{s} - \hat{c}(s+1).$$

From Table 2, the column variance is

$$\sigma_j^2 = B_j S_j^2 + \sigma^2.$$

Hence,

$$S_j^2 = \frac{\sigma_j^2 - \sigma^2}{B_j}. \tag{8}$$

With estimates of b and c, the value of B_j can be computed. Thus, the seasonal indices are obtained from (8) as

$$S_j = \sqrt{\frac{\sigma_j^2 - \sigma^2}{B_j}}.$$

If σ^2 is replaced by its expected value ($\sigma^2 = 1$), then

$$\hat{S}_j \approx \sqrt{\frac{\hat{\sigma}_j^2 - 1}{\hat{B}_j}}.$$

The estimates $\hat{S}_j$ must satisfy $\sum_{j=1}^{s}\hat{S}_j = s$. If this condition is not satisfied, they are adjusted accordingly. With the adjusted values of $\hat{S}_j$, the estimate of b is recalculated using (7) as

$$\hat{b}' = \frac{\beta}{s} - \frac{2\hat{c}}{s}\sum_{j=1}^{s} jS_j.$$

From (8), we have

$$S_1^2 = \frac{\sigma_1^2 - \sigma^2}{B_1}, \quad S_2^2 = \frac{\sigma_2^2 - \sigma^2}{B_2}, \quad \ldots, \quad S_s^2 = \frac{\sigma_s^2 - \sigma^2}{B_s}.$$

Hence,

$$\frac{S_2^2}{S_1^2} = \frac{\sigma_2^2 - \sigma^2}{\sigma_1^2 - \sigma^2} \cdot \frac{B_1}{B_2}.$$

Therefore,

$$S_2^2 = \frac{\sigma_2^2 - \sigma^2}{\sigma_1^2 - \sigma^2} \cdot \frac{B_1}{B_2} \cdot S_1^2.$$

Similarly,

$$S_3^2 = \frac{\sigma_3^2 - \sigma^2}{\sigma_2^2 - \sigma^2} \cdot \frac{B_2}{B_3} \cdot S_2^2 = \frac{\sigma_3^2 - \sigma^2}{\sigma_1^2 - \sigma^2} \cdot \frac{B_1}{B_3} \cdot S_1^2$$

$$\vdots$$

$$S_s^2 = \frac{\sigma_s^2 - \sigma^2}{\sigma_1^2 - \sigma^2} \cdot \frac{B_1}{B_s} \cdot S_1^2.$$

In general,

$$S_j^2 = \frac{\sigma_j^2 - \sigma^2}{\sigma_1^2 - \sigma^2} \cdot \frac{B_1}{B_j} \cdot S_1^2, \quad j = 2, \ldots, s.$$

Hence,

$$S_j = S_1 \sqrt{\frac{\sigma_j^2 - \sigma^2}{\sigma_1^2 - \sigma^2} \cdot \frac{B_1}{B_j}}. \tag{9}$$

Subject to the restriction $\sum_{j=1}^{s} S_j = S_1 + S_2 + \cdots + S_s = s$. From (9), it follows

$$\begin{aligned} s &= S_1 + S_1 \sqrt{\frac{\sigma_2^2 - \sigma^2}{\sigma_1^2 - \sigma^2} \cdot \frac{B_1}{B_2}} + \cdots + S_1 \sqrt{\frac{\sigma_s^2 - \sigma^2}{\sigma_1^2 - \sigma^2} \cdot \frac{B_1}{B_s}} \\ &= S_1 \left\{ 1 + \sum_{j=2}^{s} \sqrt{\frac{\sigma_j^2 - \sigma^2}{\sigma_1^2 - \sigma^2} \cdot \frac{B_1}{B_j}} \right\} \\ &= S_1 \sqrt{\frac{B_1}{\sigma_1^2 - \sigma^2}} \left\{ \sqrt{\frac{\sigma_1^2 - \sigma^2}{B_1}} + \sum_{j=2}^{s} \sqrt{\frac{\sigma_j^2 - \sigma^2}{B_j}} \right\} \\ &= S_1 \sqrt{\frac{B_1}{\sigma_1^2 - \sigma^2}} \left\{ \sum_{j=1}^{s} \sqrt{\frac{\sigma_j^2 - \sigma^2}{B_j}} \right\}. \end{aligned}$$

Thus, we have

$$S_1 = \frac{\sqrt{\frac{\sigma_1^2 - \sigma^2}{B_1}}}{\frac{1}{s} \sum_{j=1}^{s} \sqrt{\frac{\sigma_j^2 - \sigma^2}{B_j}}}.$$

Accordingly, from(9), we have

$$S_j = \sqrt{\frac{\sigma_j^2 - \sigma^2}{\sigma_1^2 - \sigma^2} \cdot \frac{B_1}{B_j}}\, S_1,$$

this becomes

$$S_j = \sqrt{\frac{B_1}{\sigma_1^2 - \sigma^2}} \cdot \frac{\sqrt{\frac{\sigma_1^2-\sigma^2}{B_1}}\sqrt{\frac{\sigma_j^2-\sigma^2}{B_j}}}{\frac{1}{s}\sum_{j=1}^{s}\sqrt{\frac{\sigma_j^2-\sigma^2}{B_j}}}.$$

Thus,

$$S_j = \frac{\sqrt{\frac{\sigma_j^2-\sigma^2}{B_j}}}{\frac{1}{s}\sum_{j=1}^{s}\sqrt{\frac{\sigma_j^2-\sigma^2}{B_j}}}, \quad j = 1, 2, \ldots, s.$$

Tables 3–5 summarize the estimators of the trend parameters and seasonal indices for both quadratic and linear trend-cycle mixed models. Table 3 corresponds to the quadratic case, while Tables 4 and 5 present the simplified expressions and estimates under the linear trend assumption.

Table 3: Summary of estimates of trend parameters and seasonal indices (see Nzenwa [11]).

Parameter	**Estimate**
a	$\alpha - \hat{b}C_1 - \hat{c}C_2$
b	$\hat{b} = \frac{\beta}{s} - 2\hat{c}C_1$
c	$\frac{\gamma}{s^2}$
S_j	$\left\{\frac{\frac{\sigma_j^2-\sigma^2}{B_j}}{\frac{1}{s}\sum_{j=1}^{s}\frac{\sigma_j^2-\sigma^2}{B_j}}\right\}^{1/2}$

When $c = 0$ in the quadratic trend-cycle component, Table 2 reduces to Table 4 obtained by Nwogu et al. [7] for the linear trend-cycle component. Consequently, Table 3 reduces to Table 5.

Table 4: Row, column and overall means and variances for the mixed model with linear trend-cycle component (see Nzenwa[11]).

Measure	**Estimate**
$\bar{X}_{i.}$	$a + bC_1 + bs(i-1) + \bar{e}_{i.}$
$\bar{X}_{.j}$	$\left\{a + \frac{(n-s)b}{2} + bj\right\} S_j + \bar{e}_{.j}$
$\bar{X}_{..}$	$a + \frac{(n-s)b}{2} + bC_1 + \bar{e}_{..}$
$\sigma_{i.}^2$	$\sigma^2 + \left[a^2\sigma_{S_j}^2 + b^2\sigma_{jS_j}^2 + 2ab\,\mathrm{Cov}(S_j, jS_j)\right] + 2s\left[ab\,\sigma_{S_j}^2 + b\,\mathrm{Cov}(S_j, jS_j)\right] \times (i-1) + s^2b^2\sigma_{S_j}^2(i-1)^2$
$\sigma_{.j}^2$	$\sigma^2 + \frac{n(n+s)\hat{b}^2}{12}S_j^2$
$\frac{n-1}{n}\sigma_x^2$	$\frac{(n^2-s^2)b^2}{12} + \left\{\left(a + \frac{(n-s)b}{2}\right)^2 + \frac{(n^2-s^2)b^2}{12}\right\}\sigma_{S_j}^2 + b^2\sigma_{jS_j}^2 + 2\left(ab + \frac{(n-s)b^2}{2}\right)\mathrm{Cov}(S_j, jS_j) + \sigma^2$

Table 5: Summary of estimates of linear trend parameters and seasonal indices (see Nzenwa [11]).

Parameter	Estimate
a	$\alpha - \hat{b}C_1$
b	$\frac{\beta}{s}$
S_j	$\left\{\frac{12(\sigma_j^2 - \sigma^2)}{n(n+s)b^2}\right\}^{1/2}$

3 Illustrative Example

This section presents an illustration of the proposed decomposition model using simulated data. A total of 100 time series were generated, each consisting of 120 observations, from the model

$$X_t = (a + bt + ct^2)S_t + e_t,$$

where $a = 5.00$, $b = -0.35$, and $c = 0.35$. The error term follows a normal distribution $e_t \sim N(0, \sigma^2)$ with $\sigma^2 = 1$, and the seasonal indices S_j are given in Table 6.

Table 6: Seasonal indices (S_j) used in the simulation of the mixed series

j	1	2	3	4	5	6	7	8	9	10	11	12
S_j	0.91	0.88	1.00	0.98	0.98	1.12	1.26	1.20	1.05	0.92	0.80	0.90

Using `MINITAB` software, the simulated series, along with the corresponding row, column, and overall means and variances of the Buys–Ballot table, were obtained. The procedure for estimating the trend parameters and seasonal indices is illustrated using one representative simulated dataset.

For this example, $m = 10$ and $s = 12$, giving $n = ms = 120$. The row means and column variances are obtained as described earlier. The regression of the row means on $(i - 1)$ is given by

$$\bar{X}_{i\cdot} = 21.21 + 50.32(i - 1) + 50.41(i - 1)^2.$$

Thus, $\alpha = 21.27$, $\beta = 50.32$, and $\gamma = 50.41$. Using these values,

$$\hat{c} = \frac{\gamma}{s^2} = \frac{50.41}{12^2} = 0.35007.$$

Since S_j is initially unknown, a trial estimate of b is obtained as

$$\hat{b} \approx \frac{\beta}{s} - \hat{c}(s + 1) = \frac{50.32}{12} - 0.35007 \times 13 = -0.35757.$$

Substituting $\hat{b}$ and $\hat{c}$ into the expression for B_j, the values of B_j are computed, as shown in Table 7. The seasonal indices are computed using

$$S_j = \sqrt{\frac{\sigma_j^2 - \sigma^2}{B_j}} \approx \sqrt{\frac{\hat{\sigma}_j^2}{B_j}}.$$

Since the sum of the estimated seasonal indices is 12.0002, they are adjusted to satisfy the constraint $\sum_{j=1}^{s} S_j = s$ using

$$\hat{S}_j = \frac{12}{12.0002} S_j.$$

Table 7: Calculation of estimates of trend parameters and seasonal indices for the mixed model

I	$\bar{X}_{i.}$	J	$\hat{\sigma}_j^2$	B_j	$\sqrt{\hat{\sigma}_j^2/B_j}$	$\hat{S}_j$	$j\hat{S}_j$	$j^2\hat{S}_j$
1	21.40	1	1714259	23525.53	0.9100	0.9100	0.9100	0.9100
2	121.70	2	1658429	24334.19	0.8800	0.8800	1.7600	3.5201
3	323.70	3	2215543	25157.56	1.0004	1.0004	3.0011	9.0033
4	625.90	4	2199010	25995.64	0.9804	0.9804	3.9217	15.6868
5	1029.10	5	2268035	26848.42	0.9798	0.9798	4.8988	24.4938
6	1532.90	6	3057041	27715.90	1.1196	1.1195	6.7172	40.3032
7	2138.10	7	3998821	28598.10	1.2605	1.2605	8.8236	61.7654
8	2843.00	8	3738172	29494.99	1.2001	1.2001	9.6006	76.8045
9	3650.00	9	2949920	30406.60	1.0500	1.0500	9.4496	85.0467
10	4557.00	10	2333928	31332.91	0.9200	0.9200	9.2001	92.0015
		11	1815853	32273.92	0.7996	0.7996	8.7955	96.7501
		12	2367801	33229.65	0.8998	0.8998	10.798	129.576
Sum					12.0002	12.0000	77.8762	635.861
Mean							6.48968	52.9884

Using the adjusted values,

$$C_1 = \frac{1}{s}\sum_{j=1}^{s} jS_j = 6.48968, \quad C_2 = \frac{1}{s}\sum_{j=1}^{s} j^2 S_j = 52.9884.$$

Thus,

$$\hat{b} = \frac{\beta}{s} - 2\hat{c}C_1 = -0.35034,$$

and

$$\hat{a} = \alpha - \hat{b}C_1 - \hat{c}C_2 = 4.99400.$$

Hence, the fitted model is

$$\hat{X}_t = (4.9940 - 0.3503t + 0.3501t^2)\hat{S}_t,$$

where $\hat{S}_t$ is given in Table 6, and the residuals are obtained as

$$\hat{e}_t = X_t - \hat{X}_t.$$

The estimates obtained for the simulated series are summarized in Table 8, where they are compared with the true parameter values used in the simulation. Due to space limitations, only ten simulation results are reported.

As shown in Table 8, the maximum absolute errors are 0.272 for the intercept (a), 0.019 for the slope (b), and 0.00 for the quadratic term (c). The seasonal effects are estimated with a very high degree of accuracy, with values closely matching those used in the simulation. The estimated error mean (μ_e) ranges from −0.192 to 0.397, while the estimated error standard deviation (σ_e) lies between 0.97 and 1.10. These results indicate that the proposed procedure produces estimates of the trend parameters that are very close to the true values, and the seasonal indices are recovered almost exactly for each season. Furthermore, the mean squared error, computed from the deviations of the estimates from their corresponding true parameter values, ranges from 0.001 to 0.011. This demonstrates that the proposed method is highly effective in recovering the underlying model parameters with a high degree of precision.

Table 8: Estimates of quadratic trend parameters and seasonal indices for series that admit the mixed model.

Parameter	X_1		X_2		X_3		X_4		X_5		X_6		X_7		X_8		X_9		X_{10}	
	Est	Err	Est	Err	Est	Err	Est	Err	Est	Err	Est	Err	Est	Err	Est	Err	Est	Err	Est	Err
$a = 5.0$	5.2722	0.272	4.9579	0.042	4.9902	0.010	4.9149	0.085	5.0283	-0.028	5.1526	0.153	5.1616	-0.162	5.1578	0.158	4.6594	0.341	4.9940	0.006
$b = -0.35$	0.3648	0.015	-0.3522	0.002	-0.3533	0.003	-0.3454	-0.005	-0.3591	0.009	-0.3648	0.015	-0.3614	0.011	-0.3586	0.009	-0.3314	-0.019	-0.3503	0.000
$c = 0.35$	0.3501	0.000	0.3500	0.000	0.3501	0.000	0.3500	0.000	0.3501	0.000	0.3501	0.000	0.3501	0.000	0.3501	0.000	0.3499	0.000	0.3501	0.000
S_1	0.9102	0.000	0.9101	0.000	0.9102	0.000	0.9097	0.000	0.9098	0.000	0.9098	0.000	0.9097	0.000	0.9100	0.000	0.9099	0.000	0.9100	0.000
S_2	0.8802	0.000	0.8798	0.000	0.8794	0.001	0.8802	0.000	0.8798	0.000	0.8800	0.000	0.8797	0.000	0.8802	0.000	0.8797	0.000	0.8800	0.000
S_3	0.9998	0.000	1.0000	0.000	0.9998	0.000	1.0000	0.000	0.9997	0.000	1.0005	0.001	1.0003	0.000	1.0001	0.000	0.9999	0.000	1.0004	0.000
S_4	0.9800	0.000	0.9800	0.000	0.9799	0.000	0.9801	0.000	0.9804	0.000	0.9800	0.000	0.9800	0.000	0.9802	0.000	0.9799	0.000	0.9804	0.000
S_5	0.9799	0.000	0.9800	0.000	0.9798	0.000	0.9803	0.000	0.9801	0.000	0.9800	0.000	0.9803	0.000	0.9804	0.000	0.9801	0.000	0.9798	0.000
S_6	1.1201	0.000	1.1199	0.000	1.1203	0.000	1.1200	0.000	1.1196	0.000	1.1198	0.000	1.1203	0.000	1.1201	0.000	1.1200	0.000	1.1195	0.000
S_7	1.2598	0.000	1.2599	0.000	1.2598	0.000	1.2603	0.000	1.2604	0.000	1.2600	0.000	1.2603	0.000	1.2596	0.000	1.2602	0.000	1.2605	-0.001
S_8	1.1999	0.000	1.2003	0.000	1.2003	0.000	1.1996	0.000	1.1999	0.000	1.1999	0.000	1.1999	0.000	1.1997	0.000	1.1999	0.000	1.2001	0.000
S_9	1.0500	0.000	1.0498	0.000	1.0501	0.000	1.0504	0.000	1.0501	0.000	1.0500	0.000	1.0500	0.000	1.0498	0.000	1.0502	0.000	1.0500	0.000
S_{10}	0.9197	0.000	0.9199	0.000	0.9202	0.000	0.9200	0.000	0.9204	0.000	0.9200	0.000	0.9200	0.000	0.9201	0.000	0.9199	0.000	0.9200	0.000
S_{11}	0.8003	0.000	0.8001	0.000	0.7999	0.000	0.7997	0.000	0.7999	0.000	0.8002	0.000	0.8000	0.000	0.7999	0.000	0.8002	0.000	0.7996	0.000
S_{12}	0.9001	0.000	0.9001	0.000	0.9001	0.000	0.8999	0.000	0.8999	0.000	0.8998	0.000	0.8997	0.000	0.8997	0.000	0.9001	0.000	0.8998	0.000
μ_e	0.0140	0.014	0.1010	-0.101	-0.1401	0.140	-0.1123	0.112	0.1350	-0.135	0.0340	0.034	-0.1177	0.118	0.0958	0.096	-0.0697	0.070	-0.1450	0.145
σ_e	0.9755	0.024	0.9658	0.034	1.0418	-0.042	1.0741	-0.074	1.1040	-0.104	1.0742	0.074	1.0201	-0.020	1.0173	0.017	0.9562	0.044	1.2070	-0.207
MAE		0.004		0.001		0.001		0.001		0.002		0.002		0.002		0.002		0.007		0.004
MSE		0.019		0.011		0.012		0.016		0.016		0.016		0.018		0.017		0.028		0.021

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4. Discussion

This study developed a decomposition procedure for mixed time series models with quadratic trend-cycle components using the Buys–Ballot table. The findings show that the proposed method effectively overcomes the limitations of conventional additive and multiplicative decomposition techniques, which are unsuitable for mixed models because they do not adequately isolate the irregular component.

The empirical simulation results demonstrate that the method provides highly accurate estimates of both trend parameters and seasonal indices. The maximum absolute errors for the estimated trend parameters were very small, while the seasonal effects were recovered almost exactly. This confirms the robustness and precision of the proposed procedure.

A major contribution of this study is the extension of existing Buys–Ballot decomposition methods beyond linear trend mixed models to quadratic trend-cycle structures. This fills an important gap in descriptive time series analysis and provides a more flexible framework for handling nonlinear seasonal data.

The low mean square error values and stable residual behavior further support the adequacy of the model. Therefore, the proposed approach is suitable for practical applications in fields such as economics, business forecasting, environmental studies, and telecommunications, where mixed seasonal time series frequently occur.

5. Conclusion

This paper discussed the procedure for the decomposition of observed time series data that admit the mixed model. It has been observed that the residual series for the mixed model cannot be obtained by the usual practice of subtraction (for the additive model) or division (for the multiplicative model) of the estimates of trend parameters and seasonal indices from the observed series. This study extends Buys–Ballot decomposition theory to quadratic mixed models, provides a formal residual derivation under mixed interaction, demonstrates empirical adequacy using randomness diagnostics, and offers a recommended decomposition framework for nonlinear seasonal time series. The procedure proposed in this study obtains the estimates of the trend parameters and seasonal indices from the row, column, and overall means and variances of the observed series presented in a Buys–Ballot table. Thereafter, the residuals are obtained by subtracting the joint estimates of the trend parameters and seasonal indices from the observed series. Empirical examples based on simulated series were used to illustrate the procedure. The results show that the procedure recovers estimates of the quadratic term and seasonal indices almost precisely, while estimates of the other trend parameters (a and b), error means, and standard deviations are recovered with a high degree of precision. It has therefore been recommended that, when a series admits the mixed model, the procedure proposed in this study be adopted in the decomposition.

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Data Availability

The dataset presented in this article is not readily available. Request to access the dataset should be directed to the corresponding author.

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Author Contributions

Conceptualization, Nwogu E.C; Methodology, Nwogu E. C and Iwueze S.I; Software, Ikediuwa U. C; Validation, Nwogu E.C and Ikediuwa U.C; Formal Analysis, Nwogu E.C; Investigation, Iwueze S.I; Resources, Ikediuwa U.C; Writing–Original Draft Preparation Nwogu E.C; Writing Review and Editing, Ikediuwa U.C; Visualization, Ikediuwa U.C; Supervision, Nwogu E.C; Project Administration, . All authors have read and agreed to the published version of the manuscript.

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