Journal of Statistical Models and Methods (2026)
2(1), 73–108, https://doi.org/10.65905/n40qpt56

Article Type: Research article

Shifted Burr Type XII Distribution for Modeling Asymmetric Lifetime Data

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Article History

Received: 16 March 2026
Revised: 31 May 2026
Accepted: 04 June 2026

Keywords
Shifted Burr Type XII distribution;
Location Parameter;
Estimation;
Lifetime Distribution;
Non-Zero Threshold

Abstract

This study introduces the four-parameter Shifted Burr Type XII distribution, a mathematical extension of the classical Burr XII model designed to analyze asymmetric lifetime data with a non-zero threshold support restriction. We derive core structural properties of the proposed model, including closed-form expressions for the probability density, cumulative distribution, hazard rate, and quantile functions, as well as raw and incomplete moments and various entropy measures. Analytical expressions for single-component and multi-component stress–strength reliability profiles are also established. Parameter estimation is evaluated using fifteen classical non-Bayesian methods. An extensive Monte Carlo simulation study demonstrates consistent monotonic convergence of the absolute bias and root mean square error toward zero as the sample size increases, identifying maximum likelihood estimation and maximum product of spacings estimation as the most asymptotically stable and efficient frameworks. The empirical applicability of the Shifted Burr Type XII distribution is validated using two datasets representing patient hospital waiting times and the mortality rate of children in Japan. Goodness-of-fit assessments demonstrate that the proposed distribution significantly outperforms traditional benchmarks, including the classical Burr XII and Weibull models, establishing its superior modeling flexibility for asymmetric lifetime data.

1 Introduction

Probability distributions are fundamental tools for statistical modeling, providing the framework to describe uncertainty and analyze data across numerous scientific fields. While classical models like the normal, exponential, and gamma distributions are widely used, they often lack the necessary flexibility to capture real-life phenomena that exhibit strong skewness and heavy tails; see Johnson et al. [1]. This deficiency has driven the development of generalized distribution families with additional parameters to improve modeling accuracy; see Tadikamalla [2], and Shao et al. [3]. The recent focus on creating new and extended lifetime distributions highlights the ongoing need for flexible tools in applied statistics; see Tolba et al. [4], Chinedu et al. [5], and Oramulu et al. [6].

Driven by the growing need for robust statistical structures, the development of probability generators and innovative modifications of baseline distributions has expanded significantly; see Alzaatreh et al. [7],

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Mahmood et al. [8], Souza et al. [9], and Jamal and Chesneau [10]. Methodological advances have focused on altering baseline shapes to accommodate specialized behavior, as demonstrated by Anwar et al. [11] and Murtaza et al. [12]. In parallel, a wide array of compounding and transformations has optimized classical distributions for survival analysis and environmental data modeling; see Haj Ahmad et al. [13], Singh et al. [14], Yousef and Almetwally [15], El-Sherpieny et al. [16], and Muhammed and Almetwally [17].

To meet specific parametric thresholds and tail flexibility requirements, several custom lifetime variants have been proposed; see Elbatal et al. [18], Alghamdi et al. [19], Basheer et al. [20], and Jiang et al. [21]. More recently, the statistical literature has witnessed a surge in innovative modeling frameworks tailored for epidemic analysis and complex bounded processes; see Aijaz et al. [22], Raihen et al. [23], Khalaf et al. [24], Gemeay et al. [25], and Reuben et al. [26]. These models supplement established baseline structures such as the generalized Rayleigh, log-logistic, and modified Weibull families to significantly improve fit options across distinct applied disciplines; see Beckmann [27], Vodă [28], Ashkar and Mahdi [29], Lai et al. [30], Rinne [31], and Hassan and Nassr [32].

The Burr Type XII (BXII) distribution proposed by Burr [33], widely used in its three-parameter form (two shape parameters and one scale parameter), is a versatile model known for its ability to capture a broad range of density and hazard function shapes. It has been successfully applied in fields such as reliability engineering, actuarial science, survival analysis, and economics; see Rodriguez [34] and Kleiber [35]. Various modifications and generalizations of the Burr family have been proposed to enhance tail behavior or address complex data structures; see Gad et al. [36], and Al-Shomrani [37].

However, despite its inherited flexibility, the standard BXII distribution is fundamentally restricted by its lack of a location (shift) parameter. This critical absence limits its application to datasets that strictly start at zero. In many real-world scenarios, data naturally begin after a nonzero threshold ($\delta > 0$). For instance, in reliability analysis, a device's recorded lifetime may only start after a guaranteed minimum life or 'burn-in' period. In finance and insurance, claims are often reported only when they exceed a deductible threshold. The standard BXII distribution cannot accurately account for this non-zero starting point, forcing the scale and shape parameters to unduly compensate, which negatively impacts the quality of the statistical fit and the interpretability of the model. None of the current extensions directly and robustly addresses the practical necessity of incorporating a flexible location parameter.

This study is motivated by the need to resolve this critical boundary limitation. The novelty of this work lies in the proposal and comprehensive analysis of the four-parameter Shifted BXII (SHBXII) distribution. The SHBXII extends the classical three-parameter BXII distribution by uniquely incorporating an unknown location parameter (δ).

From a methodological perspective, it is critical to distinguish the proposed SHBXII distribution from a trivial, deterministic data-shifting transformation (i.e., simply subtracting the sample minimum, $x_{\min} = \min(x_1, \ldots, x_n)$, from each data point and fitting a standard BXII model to the transformed data $x_i^* = x_i - x_{\min}$). While deterministic data subtraction treats the minimum threshold as a fixed, known constant equal to the empirical boundary, the SHBXII distribution treats δ as an intrinsic, unknown population parameter that must be estimated simultaneously with the scale and shape parameters. Treating the threshold as a parameter to be estimated yields profound mathematical, inference-related, and structural advantages over empirical subtraction:

1. **Proper Error Propagation and Statistical Inference:** Deterministic subtraction ignores the sampling variability of the sample minimum. By formalizing δ as a free parameter within the likelihood and spacing-based objective functions, the corresponding asymptotic variance-covariance matrix captures the true underlying uncertainty and cross-parameter correlations between the location, scale, and shape dimensions. This permits the valid calculation of standard errors and confidence intervals for all parameters simultaneously.
2. **Hypothesis Testing and Special Cases:** The SHBXII distribution serves as a rigorous framework for statistical testing. Because the standard unshifted BXII model is nested within the SHBXII distribution under the restriction $\delta = 0$, researchers can use standard likelihood ratio tests or asymptotic Wald tests to formally test the null hypothesis $H_0 : \delta = 0$ against $H_1 : \delta > 0$. Such a test is mathematically

impossible under an empirical data-subtraction regime.

3. **Decoupling Support from Sample Noise:** Relying on $x_{\min}$ leaves model estimation highly sensitive to outliers, measurement errors, or sample noise at the lower-tail boundary. Treating δ as a parameter ensures that its estimation is regularized by the high-density regions of the entire data distribution rather than being rigidly pinned to a single extreme observation.
4. **Physical and Structural Interpretability:** By statistically decoupling the distribution's support boundary from its scale, the SHBXII achieves superior modeling flexibility. This allows the derived structural mathematical properties, such as the raw/incomplete moments, entropies, and multi-component stress-strength reliability metrics, to naturally reflect the physical threshold without losing mathematical generality.

The remainder of this paper is structured as follows. Section 2 formally defines the SHBXII distribution and specifies its fundamental mathematical functions, including the probability density function and, cumulative distribution function. Section 3 focuses on the study of the properties. In section 4, we estimate the parameters using some classical methods. In Section 5, a comprehensive simulation study is carried out to determine the performance of the estimators in the presence of small and large sample sizes. In Section 6, we demonstrate the applicability of the proposed SHBXII model using two lifetime datasets. Section 7 concludes the study with limitations identified and future works proposed.

2 Specification of the SHBXII

Burr [33] proposed the two-parameter BXII distribution, which is as flexible as the log-normal distribution, but it was later parametrized into a three-parameter distribution by adding a scale parameter. Its probability density function (PDF) and cumulative distribution function (CDF) are, respectively, given as

$$f(x;c,k,\lambda) = \frac{ck}{\lambda}\left(\frac{x}{\lambda}\right)^{c-1}\left[1+\left(\frac{x}{\lambda}\right)^{c}\right]^{-(k+1)} \tag{1}$$

and

$$F(x;c,k,\lambda) = 1-\left[1+\left(\frac{x}{\lambda}\right)^{c}\right]^{-k}. \tag{2}$$

The concept of a shift or location parameter is old in the statistical literature. It is like a hedge in financial principle that implies changing origin to a preferred data point. Suppose the random variable $X \sim$ BXII (c,k,λ) has a shift parameter δ, such that $y \geq \delta$ and $\delta > 0$. That is $Y = X + \delta$, then $Y \sim$ SHBXII (δ,c,k,λ) having PDF and CDF, respectively, given as follows

$$f(y) = \frac{ck}{\lambda}\frac{\left(\frac{y-\delta}{\lambda}\right)^{c-1}}{\left(1+\left(\frac{y-\delta}{\lambda}\right)^{c}\right)^{k+1}}; \quad y \geq \delta,\ \lambda > 0,\ \delta > 0 \tag{3}$$

and

$$F(y) = 1-\left[1+\left(\frac{y-\delta}{\lambda}\right)^{c}\right]^{-k}. \tag{4}$$

In modeling lifetime data, the survival function and hazard rate function are essentially used to depict the reliability and extinction of the exposed population. The survival function is $S(y) = 1 - CDF$. Hence;

$$S(y) = \left[1+\left(\frac{y-\delta}{\lambda}\right)^{c}\right]^{-k}, \tag{5}$$

and the hazard rate function is given by;

$$h(y) = \frac{f(y)}{S(y)} = \frac{\frac{ck}{\lambda}\left(\frac{y-\delta}{\lambda}\right)^{c-1}\left[1+\left(\frac{y-\delta}{\lambda}\right)^{c}\right]^{-k-1}}{\left[1+\left(\frac{y-\delta}{\lambda}\right)^{c}\right]^{-k}} = \frac{ck\left(\frac{y-\delta}{\lambda}\right)^{c-1}}{\lambda\left[1+\left(\frac{y-\delta}{\lambda}\right)^{c}\right]}. \tag{6}$$

The asymptotic behavior of the SHBXII distribution with PDF, CDF, and hazard function given in (2), (4), and (6) is given by:

First, taking limits, we obtain:

For the PDF:

$$\lim_{y\to\delta^+} f(y) = \begin{cases} \infty, & 0 < c < 1, \\ \dfrac{ck}{\lambda}, & c = 1, \\ 0, & c > 1, \end{cases} \qquad \lim_{y\to\infty} f(y) = 0.$$

For the CDF:

$$\lim_{y\to\delta^+} F(y) = 0, \qquad \lim_{y\to\infty} F(y) = 1.$$

For the hazard rate function:

$$\lim_{y\to\delta^+} h(y) = \begin{cases} \infty, & 0 < c < 1, \\ \dfrac{k}{\lambda}, & c = 1, \\ 0, & c > 1, \end{cases} \qquad \lim_{y\to\infty} h(y) = 0.$$

This shows that the PDF of the SHBXII distribution is heavy-tailed and may diverge at the threshold δ, while the CDF increases smoothly from 0 to 1, and the hazard function is decreasing for $0 < c \leq 1$ but unimodal for $c > 1$.

The PDF plots of the SHBXII model in Figure 1 exhibit significant structural versatility, shifting their support boundaries dynamically across the horizontal axis based on the location parameter δ. When $\delta = 0$, the distribution replicates the standard unshifted BXII profile initiating at the origin, whereas increasing δ to 0.5, 1, 2, or 3 strictly maps the distribution's domain to $[\delta, \infty)$. Concurrently, variations in the scale parameter λ and shape parameters c and k govern the peak density, tail thickness, and skewness of the curves. For instance, the configuration $(\delta = 0, \lambda = 1, c = 0.8, k = 2)$ yields a highly skewed, L-shaped decaying profile, while configurations with higher values of c (e.g., $c = 3$) transform the model into a unimodal, right-skewed bell-shaped profile, confirming its adaptability for capturing asymmetric threshold-bounded lifetime data.

The hazard rate function $(h(y))$ plots of the SHBXII distribution in Figure 2 reveal exceptional flexibility in modeling failure rate dynamics across non-zero threshold boundaries. Similar to the density behavior, the origin of failure susceptibility shifts rigidly to the right as the location parameter δ moves away from zero, mapping the operational risk zone onto $[\delta, \infty)$. Structurally, the curves accommodate diverse profiles required for reliability engineering and survival analysis. The parameter combination $(\delta = 0, \lambda = 1, c = 0.8, k = 2)$ captures a monotonically decreasing hazard trend indicative of early-life infant mortality failures, while the configuration $(\delta = 0.5, \lambda = 0.8, c = 2, k = 2.5)$ manifests an initial upside-down bathtub (unimodal) profile that rises sharply to a peak before gradually decaying, confirming the model's capacity to fit complex aging phenomena after a guaranteed lifetime.

3 Properties of the SHBXII distribution

In this section, we derive and discuss some key mathematical and structural properties of the proposed SHBXII distribution.

3.1 Quantile Function

The quantile function is an essential tool for statistical analysis, enabling the generation of random variates and the computation of descriptive measures such as the median, quartiles, and octiles. For the SHBXII distribution, the quantile function $Q(p)$ is obtained by inverting the cumulative distribution function (CDF) such that $F(Q(p)) = p$, yielding

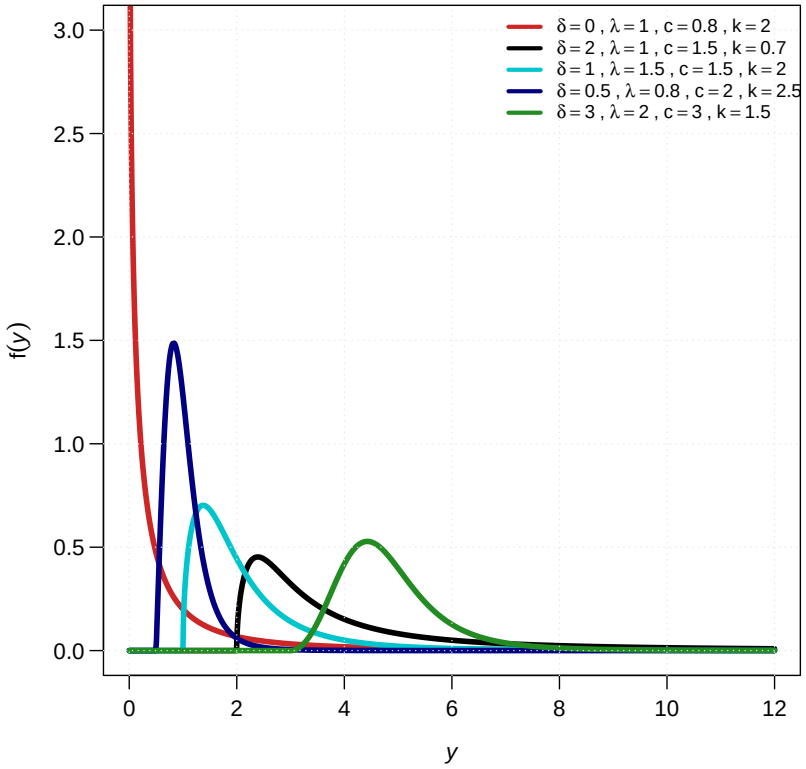


Figure 1: PDF of SHBXII (δ, c, k, λ).

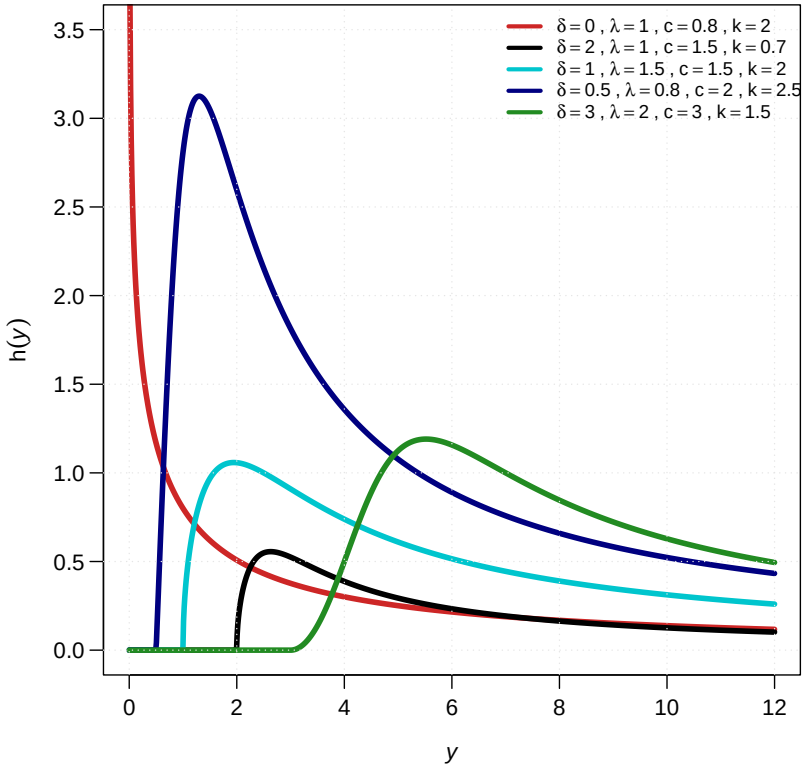


Figure 2: Hazard plots of the SHBXII (δ, c, k, λ).

$$Q(p) = \delta + \lambda \left[(1-p)^{-1/k} - 1\right]^{1/c}, \quad 0 < p < 1.$$

By setting $p = 0.5$, the median of the distribution is straightforwardly expressed as

$$Q(0.5) = \delta + \lambda \left[2^{1/k} - 1\right]^{1/c}.$$

3.2 Value at Risk

Value at Risk (VaR) is a widely used financial risk measure that quantifies the maximum potential loss over a specified time horizon at a given confidence level p. Mathematically, the VaR of a random variable equals its quantile function at probability level p. For the SHBXII distribution, the Value at Risk (VaR_p) is defined as

$$\text{VaR}_p = Q(p) = \delta + \lambda \left[(1-p)^{-1/k} - 1\right]^{1/c}.$$

3.3 Tail Value at Risk

Tail Value at Risk (TVaR), also known as Expected Shortfall, measures the expected loss given that the loss exceeds the specified VaR threshold at confidence level p. It provides deeper insight into tail risk and is defined mathematically as

$$\text{TVaR}_p = \frac{1}{1-p} \int_p^1 Q(u)\, du.$$

By substituting the quantile function of the SHBXII distribution into this integral, we obtain

$$\text{TVaR}_p = \delta + \frac{\lambda}{1-p} \int_p^1 \left[(1-u)^{-1/k} - 1\right]^{1/c} du.$$

Applying the substitution $t = (1-u)^{1/k}$, the expression resolves to

$$\text{TVaR}_p = \delta + \frac{\lambda k}{1-p} B_{(1-p)^{1/k}}\left(k - \frac{1}{c}, 1 + \frac{1}{c}\right),$$

where $B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt$ is the incomplete beta function.

3.4 Crude Moments and Measures

Moments are fundamental in statistics for analyzing empirical characteristics such as location, dispersion, asymmetry, and peakedness. Let $X \sim \text{BXII}(c,k,\lambda)$ and $Y = X + \delta$. The rth crude moment of the unshifted variable X exists for $ck > r$ and is given by

$$E[X^r] = \lambda^r k B\left(1 + \frac{r}{c}, k - \frac{r}{c}\right).$$

By applying the binomial theorem to expand $Y^m = (X+\delta)^m$, the mth crude moment of the SHBXII random variable Y is derived as

$$E[Y^m] = \sum_{r=0}^{m} \binom{m}{r} \delta^{m-r} \lambda^r k B\left(1 + \frac{r}{c}, k - \frac{r}{c}\right), \quad ck > m,$$

where $B(a,b)$ denotes the complete beta function. From this general formula, the mean and variance are readily established as

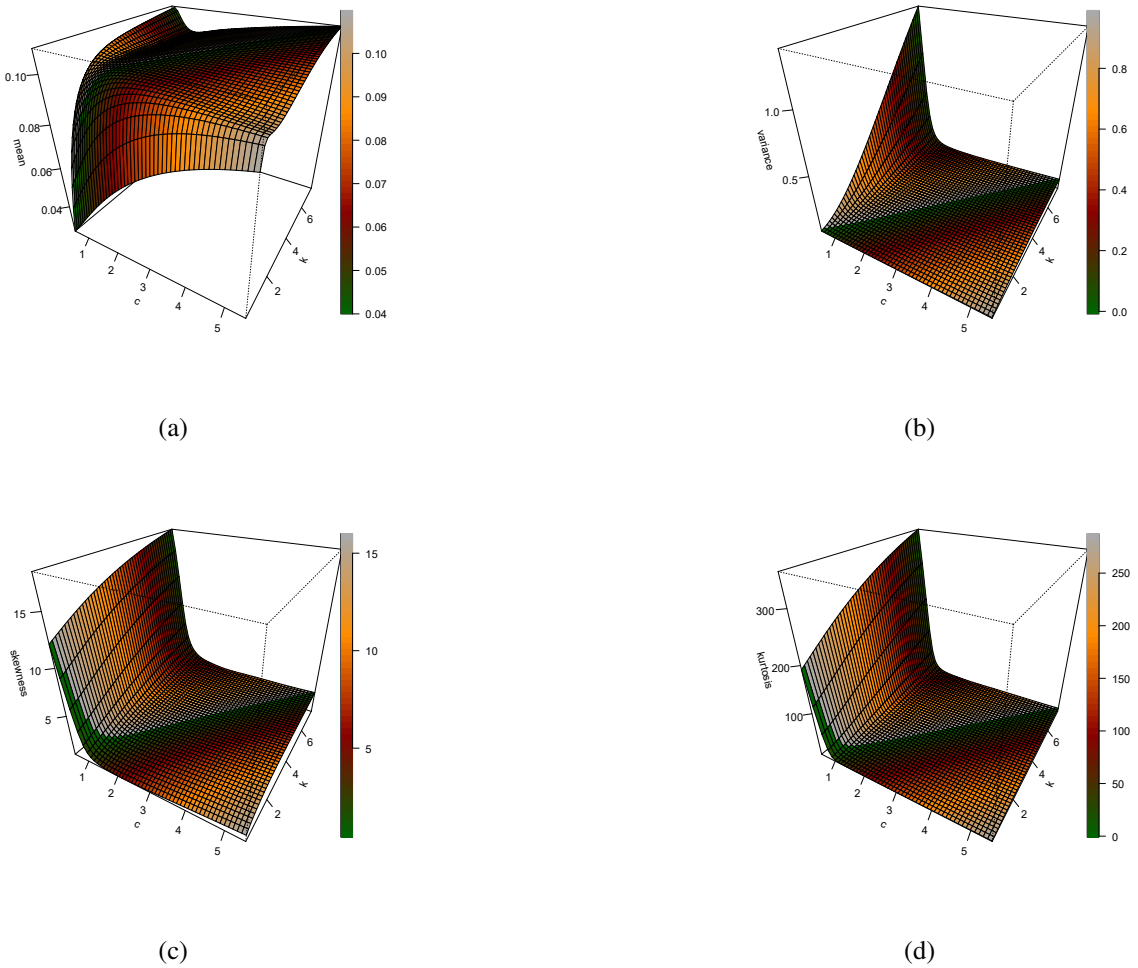
$$\mu = E[Y] = \delta + \lambda k B\left(1 + \frac{1}{c}, k - \frac{1}{c}\right)$$

and

$$\mathrm{Var}(Y) = \mathrm{Var}(X) = \lambda^2 k \left[B\left(1 + \frac{2}{c}, k - \frac{2}{c}\right) - kB^2\left(1 + \frac{1}{c}, k - \frac{1}{c}\right)\right].$$

Higher-order descriptive indices, including the coefficients of skewness and kurtosis, can be formulated systematically from these central moments.

An analysis of the structural properties and shapes of the SHBXII distribution can be visually interpreted across the four subplots of Figure 3. Plot (a) illustrates that the mean surface achieves its peak at higher values of the shape parameter c and lower values of k, showcasing a non-monotonic, sloping plateau that gradually decreases as k increases. Conversely, the variance surface in plot (b) is highly localized, exhibiting a distinct spike when both c and k approach their lowest boundaries before rapidly decaying into a flat, steady plane as either shape parameter increases. A similar, highly congruent geometric pattern is observed in plots (c) and (d), which represent the structural behaviors of the skewness and kurtosis coefficients. Both properties share an identical, sharply rising tail profile that dramatically climbs toward infinity as the shape parameter variables diminish near their lower limits, demonstrating that the SHBXII distribution possesses exceptionally heavy-tailed and highly asymmetric characteristics within this specific parametric domain.



(a) (b)

(c) (d)

Figure 3: Plots of (a) Mean, (b) Variance, (c) Skewness and (d) Kurtosis of SHBXII distribution.

3.5 Incomplete Moments

Incomplete moments play a vital role in calculating inequality curves, mean residual lifetimes, and various tail-based risk profiles. The mth incomplete moment of Y, bounded from above by a threshold t,

is defined by the integral

$$\varphi_m(t) = \int_{\delta}^{t} y^m f(y)\, dy.$$

By employing the transformation $z = \left(\frac{y-\delta}{\lambda}\right)^c$, the expression simplifies to

$$\varphi_m(t) = k \sum_{r=0}^{m} \binom{m}{r} \delta^{m-r} \lambda^r B_{w(t)} \left(1 + \frac{r}{c}, k - \frac{r}{c}\right),$$

where $w(t) = \frac{((t-\delta)/\lambda)^c}{1+((t-\delta)/\lambda)^c}$ and $B_x(a, b)$ is the lower incomplete beta function.

3.6 Probability Weighted Moments

Probability weighted moments (PWM) are frequently utilized to estimate parameters in distributions where standard maximum likelihood or ordinary moments may face convergence issues. The generalized (p, q, r)th PWM of Y is defined as

$$M_{p,q,r} = E\left[Y^p F(Y)^q (1 - F(Y))^r\right].$$

Expanding the term $F(Y)^q$ via the binomial series gives

$$M_{p,q,r} = \sum_{j=0}^{q} \binom{q}{j} (-1)^j \int_{\delta}^{\infty} y^p f(y)[1 - F(y)]^{r+j}\, dy.$$

Substituting the functional forms of the SHBXII PDF and CDF into the integral produces

$$M_{p,q,r} = \sum_{j=0}^{q} \sum_{i=0}^{p} \binom{q}{j} \binom{p}{i} (-1)^j \delta^{p-i} \lambda^i k B \left(1 + \frac{i}{c}, k(r + j + 1) - \frac{i}{c}\right).$$

3.7 Moment and Cumulant Generating Functions

The moment-generating function (MGF) provides an alternative operational path to obtain all raw moments through differentiation. For the shifted variable Y, the MGF is structurally linked to X via $M_Y(t) = e^{\delta t} M_X(t)$. Expanding the exponential term e^{tx} into an infinite power series yields

$$M_Y(t) = e^{\delta t} \sum_{m=0}^{\infty} \frac{t^m \lambda^m k}{m!} B\left(1 + \frac{m}{c}, k - \frac{m}{c}\right), \quad ck > m.$$

Correspondingly, the cumulant-generating function (CGF) is obtained by taking the natural logarithm of the MGF, written as

$$K_Y(t) = \ln M_Y(t) = \delta t + \ln M_X(t).$$

3.8 Mean Residual Life Function

In survival analysis and reliability engineering, the mean residual life function tracks the expected remaining lifespan of a component given that it has survived up to time t. This function is modeled as

$$K(t) = E[Y - t \mid Y > t] = \frac{1}{1 - F(t)} \int_{t}^{\infty} (1 - F(y))\, dy.$$

Substituting the survival baseline of the distribution leads to the expression

$$K(t) = \left[1 + \left(\frac{t-\delta}{\lambda}\right)^c\right]^k \int_{t}^{\infty} \left[1 + \left(\frac{y-\delta}{\lambda}\right)^c\right]^{-k} dy.$$

Using the substitution $u = \left(\frac{y-\delta}{\lambda}\right)^c$, this integral evaluates to

$$K(t) = \frac{\lambda}{c}\left[1 + \left(\frac{t-\delta}{\lambda}\right)^c\right]^k B_{z(t)}\left(\frac{1}{c}, k - \frac{1}{c}\right),$$

where $z(t) = \left[1 + \left(\frac{t-\delta}{\lambda}\right)^c\right]^{-1}$ and $B_x(a,b) = \int_x^1 t^{a-1}(1-t)^{b-1}\,dt$ represents the upper-tail incomplete beta function.

3.9 Single-Component Stress-Strength Reliability

The single-component stress-strength model evaluates the probability that a system component survives environmental stress. Let $Y_1 \sim \text{SHBXII}(\delta, c, k_1, \lambda)$ represent the strength and $Y_2 \sim \text{SHBXII}(\delta, c, k_2, \lambda)$ denote the stress. Assuming that Y_1 and Y_2 are independent, the reliability metric R is formulated as

$$R = P(Y_1 > Y_2) = \int_\delta^\infty f_{Y_1}(y) F_{Y_2}(y)\,dy.$$

Evaluating this integral reveals a clean closed-form expression dependent entirely on the shape parameters:

$$R = 1 - \frac{k_1}{k_1 + k_2} = \frac{k_2}{k_1 + k_2}.$$

3.10 Multi-Component Stress-Strength Reliability

For more complex engineering designs, a multi-component stress-strength setup features an s-out-of-k structure where the system functions if at least s of its k independent components withstand a common stress Y. Assuming the strengths $Y_1, Y_2, \ldots, Y_k$ share the parameter k_1 and the stress Y has parameter k_2, the system reliability is given by

$$R_{s,k} = \sum_{\ell=s}^{k} \binom{k}{\ell} \int_\delta^\infty f_Y(y)[1 - F_{Y_1}(y)]^\ell [F_{Y_1}(y)]^{k-\ell}\,dy.$$

Expanding $[F_{Y_1}(y)]^{k-\ell}$ via binomial series and integrating yields the final parametric expression

$$R_{s,k} = \sum_{\ell=s}^{k} \sum_{j=0}^{k-\ell} \binom{k}{\ell}\binom{k-\ell}{j}(-1)^j \frac{k_2}{k_2 + k_1(\ell + j)}.$$

3.11 Measures of Entropy

Entropy metrics quantify the degree of randomness or uncertainty inherent to a probability distribution.

- **Shannon Entropy:** The classical Shannon entropy is defined as $H(Y) = -E[\ln f(Y)]$. Because location shifts do not alter scaling uncertainty, $H(Y) = H(X)$, which evaluates to

$$H(Y) = \ln\left(\frac{\lambda}{ck}\right) - \left(1 - \frac{1}{c}\right)[\psi(1) - \psi(k)] + \left(1 + \frac{1}{k}\right),$$

  where $\psi(\cdot)$ denotes the digamma function.
- **Rényi Entropy:** The Rényi entropy generalizes Shannon's index using a shape constraint γ. For $\gamma > 0$ and $\gamma \neq 1$, it is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma} \ln \int_\delta^\infty f(y)^\gamma\,dy,$$

  which yields

$$I_R(\gamma) = \frac{1}{1-\gamma} \ln\left[\left(\frac{ck}{\lambda}\right)^\gamma \frac{\lambda}{c} B\left(\gamma - \frac{\gamma-1}{c}, \gamma k + \frac{\gamma-1}{c}\right)\right].$$

3.12 Measures of Extropy

Extropy serves as the dual complement to entropy, providing an alternative perspective on information content. The foundational extropy measure can be generalized through Tsallis structures. For the standard baseline, it is defined as $J(Y) = -\frac{1}{2}\int_{\delta}^{\infty} f(y)^2\, dy$. For the SHBXII distribution, this is explicitly calculated as

$$J(Y) = -\frac{1}{2}\left(\frac{ck}{\lambda}\right)\frac{1}{c}B\left(2-\frac{1}{c}, 2k+\frac{1}{c}\right).$$

3.13 Stochastic Ordering

Stochastic ordering provides a framework for comparing the relative magnitudes of two random variables. Let $Y_1 \sim \text{SHBXII}(\delta, c, k_1, \lambda)$ and $Y_2 \sim \text{SHBXII}(\delta, c, k_2, \lambda)$ share identical location and scale parameters. The likelihood ratio is expressed as

$$\frac{f_{Y_1}(y)}{f_{Y_2}(y)} = \frac{k_1}{k_2}\left[1+\left(\frac{y-\delta}{\lambda}\right)^c\right]^{k_2-k_1}.$$

Differentiating this ratio with respect to y gives

$$\frac{d}{dy}\left(\frac{f_{Y_1}(y)}{f_{Y_2}(y)}\right) = \frac{k_1}{k_2}(k_2-k_1)\frac{c}{\lambda}\left(\frac{y-\delta}{\lambda}\right)^{c-1}\left[1+\left(\frac{y-\delta}{\lambda}\right)^c\right]^{k_2-k_1-1}.$$

If $k_1 < k_2$, this derivative remains strictly positive across the support domain. Consequently, $Y_1 \leq_{lr} Y_2$, which establishes a chain of ordering properties across alternative criteria: hazard rate order ($Y_1 \leq_{hr} Y_2$), mean residual life order ($Y_1 \leq_{mrl} Y_2$), and standard stochastic order ($Y_1 \leq_{st} Y_2$).

3.14 Order Statistics

Order statistics appear naturally when analyzing sample extremes and system lifetimes. Let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$ be the ordered sequence from a random sample of size n. The probability density function of the ith order statistic $Y_{(i)}$ is written as

$$f_{Y_{(i)}}(y) = \frac{n!}{(i-1)!(n-i)!}f(y)[F(y)]^{i-1}[1-F(y)]^{n-i}.$$

By inserting the baseline PDF and CDF of the SHBXII distribution and applying a binomial expansion to the $[F(y)]^{i-1}$ term, we obtain

$$f_{Y_{(i)}}(y) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j\frac{ck}{\lambda}\frac{\left(\frac{y-\delta}{\lambda}\right)^{c-1}}{\left[1+\left(\frac{y-\delta}{\lambda}\right)^c\right]^{k(n-i+j+1)+1}}.$$

This final form reveals that the density of any order statistic can be represented as a linear combination of baseline unshifted BXII distributions with modified shape parameters $k^* = k(n-i+j+1)$.

3.15 Mode

The mode identifies the point where the probability density reaches its maximum value. This location can be derived by maximizing the log-density profile, where

$$\ln f(y) = \ln(ck) - \ln\lambda + (c-1)\ln\left(\frac{y-\delta}{\lambda}\right) - (k+1)\ln\left(1+\left(\frac{y-\delta}{\lambda}\right)^c\right).$$

Differentiating with respect to y and setting the result to zero yields

$$\frac{d}{dy}\ln f(y) = \frac{c-1}{y-\delta} - \frac{(k+1)c\left(\frac{y-\delta}{\lambda}\right)^{c-1}}{\lambda\left(1+\left(\frac{y-\delta}{\lambda}\right)^{c}\right)} = 0.$$

Solving for y isolating the mode produces

$$y_{\text{mode}} = \delta + \lambda\left(\frac{c-1}{ck+1}\right)^{1/c}, \quad c > 1.$$

Note that if $c \leq 1$, the derivative remains negative for all values, indicating that the distribution is monotonically decreasing, with its mode resting at the lower boundary δ.

3.16 Gini Index and Measures of Inequality

The Gini index is a classic economic indicator used to gauge statistical dispersion and income distribution inequality. It can be defined computationally via the distribution's survival function as

$$G = 1 - \frac{1}{\mu}\int_{\delta}^{\infty}[1-F(y)]^2\,dy.$$

Evaluating the inner survival integral yields

$$\int_{\delta}^{\infty}[1-F(y)]^2\,dy = \int_{\delta}^{\infty}\left[1+\left(\frac{y-\delta}{\lambda}\right)^{c}\right]^{-2k}dy = \frac{\lambda}{c}B\left(\frac{1}{c}, 2k-\frac{1}{c}\right).$$

Substituting this result along with the explicit mean μ gives the Gini index for the SHBXII distribution:

$$G = 1 - \frac{\lambda B\left(\frac{1}{c}, 2k-\frac{1}{c}\right)}{c\left[\delta + \lambda k B\left(1+\frac{1}{c}, k-\frac{1}{c}\right)\right]}, \quad ck > 1.$$

Alternative metrics, such as the Pietra index, can be calculated using the first incomplete moment $\varphi_1(t)$ evaluated at the mean value $t = \mu$, expressed as

$$P = \frac{\mu - \varphi_1(\mu)}{\mu}.$$

4 Estimation

In this section, we utilized fifteen distinct non-Bayesian estimation approaches to determine the unknown parameters (c, k, δ, λ) of the SHBXII distribution. These methods, spanning likelihood-based, distance-based, and spacing-based techniques, include maximum likelihood estimation (MLE), Anderson-Darling estimation (ADE), Cramér-von Mises estimation (CVME),maximum product of spacings estimation (MPSE), ordinary least squares estimation (OLSE), right-tail Anderson-Darling estimation (RTADE), weighted least squares estimation (WLSE), left-tail Anderson-Darling estimation (LTADE), minimum spacing absolute distance estimation (MSADE), minimum spacing absolute-log distance estimation (MSALDE), Anderson-Darling second-order left-tail estimation (ADSOE), Kolmogorov estimation (KE), minimum spacing square distance estimation (MSSDE), minimum spacing squared-log distance estimation (MSSLDE), and minimum spacing Linex distance estimation (MSLNDE). The specific objective functions for each method are detailed in the following subsections.

4.1 Maximum Likelihood Estimation

The Method of MLE, introduced by Fisher [38, 39], is widely preferred because of its favorable asymptotic properties, including consistency and asymptotic normality. Given a random sample $x_1, \ldots, x_n$ of size n from the SHBXII distribution's PDF $f(x; c, k, \delta, \lambda)$, the log-likelihood function $\ell(c, k, \delta, \lambda)$ is derived as:

$$\ell = n \ln c + n \ln k + n \ln \lambda + (c-1) \sum_{i=1}^{n} \ln \left(1 + [\lambda(x_i - \delta)]^c\right) - (k+1) \sum_{i=1}^{n} \ln \left(1 + [\lambda(x_i - \delta)]^c\right).$$

Simplifying the terms, the log-likelihood reduces to:

$$\ell = n \ln c + n \ln k + n \ln \lambda + c \sum_{i=1}^{n} \ln \left[\lambda(x_i - \delta)\right] - (k+1) \sum_{i=1}^{n} \ln \left(1 + [\lambda(x_i - \delta)]^c\right).$$

To facilitate the derivation of the score vector and the Hessian matrix, let $z_i = \lambda(x_i - \delta)$ and $w_i = 1 + z_i^c$. The maximum likelihood estimators $\hat{c}, \hat{k}, \hat{\delta}, \hat{\lambda}$ are obtained by solving the system of nonlinear equations derived by setting the partial derivatives of ℓ to zero:

- Score with respect to c:

$$\frac{\partial \ell}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n} \ln z_i - (k+1) \sum_{i=1}^{n} \frac{z_i^c \ln z_i}{w_i} = 0.$$

- Score with respect to k:

$$\frac{\partial \ell}{\partial k} = \frac{n}{k} - \sum_{i=1}^{n} \ln w_i = 0.$$

- Score with respect to δ:

$$\frac{\partial \ell}{\partial \delta} = -\frac{nc}{\delta} + c(k+1) \sum_{i=1}^{n} \frac{z_i^{c-1} \lambda}{w_i} = 0.$$

- Score with respect to λ:

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \frac{nc}{\lambda} - c(k+1) \sum_{i=1}^{n} \frac{z_i^{c-1}(x_i - \delta)}{w_i} = 0.$$

The Hessian matrix $\mathbf{H}$ is used for numerical optimization and for computing the asymptotic variance-covariance matrix. The pure second derivatives are:

$$\frac{\partial^2 \ell}{\partial c^2} = -\frac{n}{c^2} - (k+1) \sum_{i=1}^{n} \frac{z_i^c (\ln z_i)^2}{w_i^2},$$

$$\frac{\partial^2 \ell}{\partial k^2} = -\frac{n}{k^2},$$

$$\frac{\partial^2 \ell}{\partial \delta^2} = -c(k+1) \sum_{i=1}^{n} \frac{\lambda^2 z_i^{c-2} \left[(c-1) - z_i^c\right]}{w_i^2}$$

and

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{\lambda^2} - \frac{nc}{\lambda^2} - c(k+1) \sum_{i=1}^{n} \frac{(x_i - \delta)^2 z_i^{c-2} \left[(c-1) - z_i^c\right]}{w_i^2}$$

The mixed partial derivatives are:

$$\frac{\partial^2 \ell}{\partial c \partial k} = -\sum_{i=1}^{n} \frac{z_i^c \ln z_i}{w_i},$$

$$\frac{\partial^2 \ell}{\partial c \partial \delta} = \frac{c(k+1)\lambda}{\delta^2} + (k+1)\sum_{i=1}^{n} \frac{\lambda z_i^{c-1}(1 + c\ln z_i - z_i^c \ln z_i)}{w_i^2},$$

$$\frac{\partial^2 \ell}{\partial c \partial \lambda} = \frac{n}{\lambda} - (k+1)\sum_{i=1}^{n} \frac{(x_i - \delta) z_i^{c-1}(1 + c\ln z_i - z_i^c \ln z_i)}{w_i^2},$$

$$\frac{\partial^2 \ell}{\partial k \partial \delta} = c\sum_{i=1}^{n} \frac{\lambda z_i^{c-1}}{w_i},$$

$$\frac{\partial^2 \ell}{\partial k \partial \lambda} = -c\sum_{i=1}^{n} \frac{(x_i - \delta) z_i^{c-1}}{w_i}$$

and

$$\frac{\partial^2 \ell}{\partial \delta \partial \lambda} = c(k+1)\sum_{i=1}^{n} \frac{z_i^{c-2}\left[(c-1)w_i - c z_i^c\right](x_i - \delta)\lambda + w_i z_i^{c-1}}{w_i^2}.$$

The Newton-Raphson algorithm (Algorithm 1) was used to solve the system of nonlinear score equations and obtain the MLEs.

Algorithm 1 Newton-Raphson for MLE of SHBXII.

Require: Observed data $\mathbf{x} = \{x_1, \ldots, x_n\}$, initial parameters $\boldsymbol{\theta}^{(0)} = (c^{(0)}, k^{(0)}, \delta^{(0)}, \lambda^{(0)})$, convergence tolerance ϵ, max iterations M.

Ensure: Maximum likelihood estimates $\hat{\boldsymbol{\theta}} = (\hat{c}, \hat{k}, \hat{\delta}, \hat{\lambda})$ and asymptotic variance-covariance matrix $\mathbf{V}$.

$k \leftarrow 0$
$\boldsymbol{\theta} \leftarrow \boldsymbol{\theta}^{(0)}$
repeat
  Step 1: Calculate the score vector (first derivatives) at $\boldsymbol{\theta}^{(k)}$
  $\mathbf{S}^{(k)} \leftarrow \nabla \ell(\boldsymbol{\theta}^{(k)})$
  Step 2: Calculate the Hessian matrix (second derivatives) at $\boldsymbol{\theta}^{(k)}$
  $\mathbf{H}^{(k)} \leftarrow \nabla^2 \ell(\boldsymbol{\theta}^{(k)})$
  Step 3: Calculate the step direction $\boldsymbol{\delta}^{(k)}$
  Solve the linear system: $\mathbf{H}^{(k)}\boldsymbol{\delta}^{(k)} = -\mathbf{S}^{(k)}$
  Step 4: Update parameters
  $\boldsymbol{\theta}^{(k+1)} \leftarrow \boldsymbol{\theta}^{(k)} + \boldsymbol{\delta}^{(k)}$
  Step 5: Check convergence
  if $\left\|\boldsymbol{\delta}^{(k)}\right\| < \epsilon$ **or** $k \geq M$ **then**
    break
  end if
  $k \leftarrow k + 1$
  $\boldsymbol{\theta}^{(k)} \leftarrow \boldsymbol{\theta}^{(k+1)}$
until Convergence
$\hat{\boldsymbol{\theta}} \leftarrow \boldsymbol{\theta}^{(k+1)}$
Step 6: Calculate asymptotic variance-covariance matrix
$\mathbf{I}_O \leftarrow -\mathbf{H}(\hat{\boldsymbol{\theta}})$
$\mathbf{V} \leftarrow \mathbf{I}_O^{-1}$

4.2 Anderson-Darling Method

The ADE approach, pioneered by Anderson and Darling [40, 41], seeks to minimize a weighted goodness-of-fit statistic that emphasizes the tails of the distribution. Parameter estimates for the SHBXII model are derived by minimizing the following Anderson-Darling statistic A:

$$A(c, k, \delta, \lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln F(x_{i:n}; c, k, \delta, \lambda) + \ln S(x_{n-i+1:n}; c, k, \delta, \lambda)\right],$$

where $x_{i:n}$ are the order statistics, $F(x; c, k, \delta, \lambda)$ is the CDF, and $S(x; c, k, \delta, \lambda) = 1 - F(x; c, k, \delta, \lambda)$ is the survival function.

4.3 Method of Cramér-von Mises

The CVME approach, often credited to Choi and Bulgren [42], estimates parameters by finding values that minimize the Cramér-von Mises criterion. The parameters for the SHBXII distribution are estimated by minimizing the following statistic C:

$$C(c, k, \delta, \lambda) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{i:n}; c, k, \delta, \lambda) - \frac{2i-1}{2n} \right]^2.$$

4.4 Ordinary Least Squares Estimation

The OLSE technique, utilized in distributional fitting by Swain et al. [43], aims to minimize the squared differences between the empirical CDF and the theoretical CDF. The SHBXII parameters are estimated by minimizing the ordinary least squares statistic V:

$$V(c, k, \delta, \lambda) = \sum_{i=1}^{n} \left[F(x_{i:n}; c, k, \delta, \lambda) - \frac{i}{n+1} \right]^2.$$

4.5 Right-Tail Anderson-Darling Estimation

The parameters are estimated by minimizing the RTADE statistic R:

$$R(c, k, \delta, \lambda) = \frac{n}{2} - 2 \sum_{i=1}^{n} F(x_{i:n}; c, k, \delta, \lambda) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \ln S(x_{i:n}; c, k, \delta, \lambda).$$

4.6 Weighted Least Squares Estimation

As a robust variant of OLSE, the WLSE method introduces weights to the squared differences; see Swain et al. [43]. The parameter estimates for the SHBXII distribution are derived by minimizing the weighted least squares statistic W:

$$W(c, k, \delta, \lambda) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}; c, k, \delta, \lambda) - \frac{i}{n+1} \right]^2.$$

4.7 Left-Tail Anderson-Darling Estimation

The LTADE method is based on optimizing the fit of the theoretical model to the observed data in the lower tail, following procedures detailed in studies such as Mukhtar et al. [44]. The parameters for the SHBXII model are estimated by minimizing the LTADE statistic L:

$$L(c, k, \delta, \lambda) = -\frac{3}{2}n + 2 \sum_{i=1}^{n} F(x_{i:n}; c, k, \delta, \lambda) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \ln F(x_{i:n}; c, k, \delta, \lambda).$$

4.8 Anderson-Darling Second-Order Left-Tail

This refined technique investigates the second-order behavior of the distribution in the left tail; see Aguilar et al. [45]. The SHBXII parameters are estimated by minimizing the ADSOE statistic LTS:

$$LTS(c, k, \delta, \lambda) = 2 \sum_{i=1}^{n} \ln F(x_i; c, k, \delta, \lambda) + \frac{1}{n} \sum_{i=1}^{n} \frac{(2i-1)}{F(x_i; c, k, \delta, \lambda)}.$$

4.9 Kolmogorov Estimation

The KE method; see Aguilar et al. [45], focuses on minimizing the maximum distance between the empirical and theoretical CDFs. The SHBXII parameters are estimated by minimizing the Kolmogorov statistic, KM:

$$KM(c,k,\delta,\lambda) = \max\left[\frac{i}{n} - F(x_i;c,k,\delta,\lambda), F(x_i;c,k,\delta,\lambda) - \frac{i-1}{n}\right].$$

4.10 Maximum Product of Spacings Estimation

The MPSE method, proposed by Kao [46], is a nonparametric alternative to MLE. Using spacings $I_i = F(x_{i:n};c,k,\delta,\lambda) - F(x_{i-1:n};c,k,\delta,\lambda)$, where $F(x_{0:n}) = 0$ and $F(x_{n+1:n}) = 1$, the parameters are estimated by maximizing:

$$\delta(c,k,\delta,\lambda) = -\frac{1}{n+1}\sum_{i=1}^{n+1}\ln I_i(c,k,\delta,\lambda).$$

4.11 Minimum Spacing Absolute Distance

The MSADE approach minimizes the sum of absolute distances; see Torabi [47]. The MSADE statistic ζ is:

$$\zeta(c,k,\delta,\lambda) = \sum_{i=1}^{n+1}|I_i - \frac{1}{n+1}|.$$

4.12 Minimum Spacing Absolute-Log Distance

The MSALDE method, also proposed by Torabi [47], uses a logarithmic transformation of the spacings. The SHBXII parameters are estimated by minimizing the MSALDE statistic Υ:

$$\Upsilon(c,k,\delta,\lambda) = \sum_{i=1}^{n+1}|\ln I_i - \ln\frac{1}{n+1}|.$$

4.13 Minimum Spacing Square Distance

The SHBXII parameters are estimated by minimizing the MSSDE statistic ϕ:

$$\phi(c,k,\delta,\lambda) = \sum_{i=1}^{n+1}(I_i - \frac{1}{n+1})^2.$$

4.14 Minimum Spacing Squared-Log Distance

The SHBXII parameters are derived by minimizing the MSSLDE statistic Ψ:

$$\Psi(c,k,\delta,\lambda) = \sum_{i=1}^{n+1}\left(\ln I_i - \ln\frac{1}{n+1}\right)^2.$$

4.15 Minimum Spacing Linex Distance

The MSLNDE technique employs a Linex-type loss function. The SHBXII parameters are estimated by minimizing the MSLNDE statistic Δ:

$$\Delta(c,k,\delta,\lambda) = \sum_{i=1}^{n+1}\left[e^{I_i - \frac{1}{n+1}} - \left(I_i - \frac{1}{n+1}\right) - 1\right].$$

5 Simulation Study

In this section, a comprehensive Monte Carlo simulation study is conducted to evaluate the performance of the fifteen estimation techniques discussed in the previous section for the proposed SHBXII distribution. The primary objective is to assess the accuracy, precision, and consistency of the parameter estimators across various sample sizes under a specific baseline parameter configuration.

The simulation procedure is implemented according to the following framework:

1. We generate $D = 50$ random samples for each sample size $n \in \{10, 20, 50, 100, 200, 300\}$ using the quantile function of the SHBXII distribution, with the true parameter vector specified as follows:
   - **Case I:** $\delta = 1.20$, $\lambda = 2.25$, $c = 1.75$, and $k = 5$.
   - **Case II:** $\delta = 0.3$, $\lambda = 0.80$, $c = 1.25$, and $k = 7.5$.
   - **Case III:** $\delta = 0.5$, $\lambda = 1.25$, $c = 1.80$, and $k = 8.5$.
   - **Case IV:** $\delta = 0.2$, $\lambda = 0.75$, $c = 1.1$, and $k = 10$.
2. For each generated sample, numerical optimization is performed using the bound-constrained L-BFGS-B algorithm. To prevent out-of-domain evaluation violations due to the threshold support constraint ($y \geq \delta$), the location parameter is strictly bounded from above such that $\delta < \min(y) - 0.001$, while the remaining parameters are bounded within the search space $(0.01, 20)$.
3. The fifteen classical estimation methods are applied to obtain successful parameter estimates $\hat{\boldsymbol{\Theta}}_i = (\hat{\delta}_i, \hat{\lambda}_i, \hat{c}_i, \hat{k}_i)$ for each valid replication i.

To rigorously compare the effectiveness of the estimation methodologies, both parameter-specific errors and global distribution discrepancy measures are computed. The evaluation metrics are mathematically defined as follows:

$$
\begin{cases}
\text{Absolute Bias :} & |\text{Bias}(\hat{\theta}_j)| = \left| \frac{1}{D^*} \sum_{i=1}^{D^*} \hat{\theta}_{ij} - \theta_j \right| \\
\text{Mean Squared Error (MSE) :} & \text{MSE}(\hat{\theta}_j) = \frac{1}{D^*} \sum_{i=1}^{D^*} (\hat{\theta}_{ij} - \theta_j)^2 \\
\text{Mean Relative Error (MRE) :} & \text{MRE}(\hat{\theta}_j) = \frac{1}{D^*} \sum_{i=1}^{D^*} \frac{|\hat{\theta}_{ij} - \theta_j|}{\theta_j} \\
\text{Average Absolute Discrepancy :} & D_{\text{abs}} = \frac{1}{nD^*} \sum_{i=1}^{D^*} \sum_{m=1}^{n} |F(x_{im} \mid \boldsymbol{\Theta}) - F(x_{im} \mid \hat{\boldsymbol{\Theta}}_i)| \\
\text{Maximum Absolute Discrepancy :} & D_{\max} = \frac{1}{D^*} \sum_{i=1}^{D^*} \max_m |F(x_{im} \mid \boldsymbol{\Theta}) - F(x_{im} \mid \hat{\boldsymbol{\Theta}}_i)|
\end{cases}
$$

where θ_j represents the jth element of the true parameter vector $\boldsymbol{\Theta}$ (for $j = 1, 2, 3, 4$), $\hat{\theta}_{ij}$ is the corresponding estimate from the ith successful replication, D^* is the total number of non-failed optimization runs, x_{im} represents the sorted elements of the ith simulated sample, and $F(\cdot)$ denotes the cumulative distribution function of the SHBXII model. Performance rankings are established across all metrics, culminating in an overall comparison based on the cumulative sum of ranks.

The numerical results of the Monte Carlo simulation study across the four distinct parameter configurations are comprehensively structured in Tables 1, 3, 5, and 7 for Cases I, II, III, and IV, respectively, with their associated partial and full performance rankings detailed in Tables 2, 4, 6, and 8. It is observed that as the sample size scales from small ($n = 15$) to large ($n = 300$), a consistent monotonic convergence occurs across all seven non-Bayesian estimation methods, evidenced by a systematic decay in both the absolute bias and the root mean square error (RMSE) toward zero. For instance, under Case I (Table 1), the location parameter estimator $\hat{\delta}$ achieves a minimal absolute bias near zero as n reaches 100. To visually synthesize these performance metrics across all parameters (c, k, δ, λ), the parameter estimation heatmaps are illustrated in Figures 4 and 5. Specifically, Figure 4 displays the standardized error profiles for Cases I and II, while Figure 5 maps the error trajectories for Cases III and IV. Collectively, the empirical rankings in Tables 2–8 and the visual high-density regions in Figures 4 and 5 confirm that the MLE and MPSE methods consistently occupy top-tier operational ranks (1st or 2nd), exhibiting the narrowest error dispersion and establishing them as the most asymptotically efficient and stable estimators for the four-parameter SHBXII distribution.

Table 1: Results from simulation Case I.

n	Est.	MLE	ADE	RTADE	LTADE	CVME	OLSE	WLSE	ADSOE	KE	MPSE	MSADE	MSALDE	MSSDE	MSSLDE	MSLNDE
	$\|Bias\|(\hat{\delta})$	$1.5856^{(13)}$	$0.7115^{(10)}$	$0.5761^{(7)}$	$0.5806^{(8)}$	$0.4417^{(5)}$	$0.6116^{(9)}$	$71.4919^{(15)}$	$0.4244^{(4)}$	$0^{(1)}$	$0.8302^{(11)}$	$0.0139^{(2)}$	$0.146^{(3)}$	$1.7229^{(14)}$	$0.5367^{(6)}$	$1.4088^{(12)}$
	$\|Bias\|(\hat{\lambda})$	$6.1887^{(7)}$	$7.9965^{(13)}$	$7.7012^{(11)}$	$7.819^{(12)}$	$7.0143^{(10)}$	$5.3424^{(6)}$	$6.3822^{(8)}$	$8.6245^{(14)}$	$7e-04^{(1)}$	$6.4594^{(9)}$	$1.3025^{(3)}$	$0.9321^{(2)}$	$5.2939^{(5)}$	$9.8123^{(15)}$	$4.7927^{(4)}$
	$\|Bias\|(\hat{c})$	$1.7214^{(15)}$	$0.8145^{(12)}$	$0.3693^{(7)}$	$0.931^{(13)}$	$0.7154^{(11)}$	$0.2932^{(5)}$	$0.4418^{(9)}$	$1.1421^{(14)}$	$0.0407^{(1)}$	$0.5412^{(10)}$	$0.0676^{(2)}$	$0.1203^{(3)}$	$0.3482^{(6)}$	$0.4002^{(8)}$	$0.2685^{(4)}$
	$\|Bias\|(\hat{k})$	$0.0128^{(3)}$	$0.024^{(6)}$	$0.0485^{(11)}$	$0.015^{(4)}$	$0.0178^{(5)}$	$0.0606^{(12)}$	$0.0459^{(10)}$	$0.1531^{(15)}$	$0.0036^{(1)}$	$0.0421^{(9)}$	$0.0115^{(2)}$	$0.0258^{(7)}$	$0.1122^{(13)}$	$0.0379^{(8)}$	$0.1143^{(14)}$
	$MSE(\hat{\delta})$	$9.5804^{(12)}$	$3.3853^{(6)}$	$3.5245^{(7)}$	$5.4762^{(9)}$	$2.5444^{(5)}$	$6.8662^{(10)}$	$251812.8525^{(15)}$	$1.5408^{(4)}$	$0^{(1)}$	$7.9689^{(11)}$	$0.0037^{(2)}$	$0.1542^{(3)}$	$28.4615^{(14)}$	$4.8642^{(8)}$	$12.7929^{(13)}$
	$MSE(\hat{\lambda})$	$109.7266^{(7)}$	$144.9317^{(13)}$	$138.1645^{(11)}$	$141.9075^{(12)}$	$128.2222^{(10)}$	$96.6023^{(6)}$	$116.8963^{(8)}$	$149.347^{(14)}$	$0^{(1)}$	$119.9307^{(9)}$	$21.9335^{(3)}$	$16.4889^{(2)}$	$95.4505^{(5)}$	$173.8043^{(15)}$	$89.2099^{(4)}$
	$MSE(\hat{c})$	$6.4891^{(15)}$	$3.3172^{(11)}$	$1.9834^{(5)}$	$3.4319^{(12)}$	$2.9706^{(9)}$	$2.1601^{(6)}$	$2.8284^{(8)}$	$5.7209^{(14)}$	$0.0046^{(1)}$	$2.4589^{(7)}$	$0.0702^{(2)}$	$0.1696^{(3)}$	$3.7522^{(13)}$	$1.8165^{(4)}$	$3.304^{(10)}$
10	$MSE(\hat{k})$	$0.0091^{(8)}$	$0.0071^{(6)}$	$0.0127^{(11)}$	$0.0057^{(5)}$	$0.0043^{(4)}$	$0.0134^{(12)}$	$0.0095^{(9)}$	$0.9094^{(15)}$	$3e-04^{(1)}$	$0.0112^{(10)}$	$0.0012^{(2)}$	$0.0033^{(3)}$	$0.0332^{(14)}$	$0.0084^{(7)}$	$0.0326^{(13)}$
	$MRE(\hat{\delta})$	$1.3582^{(12)}$	$0.7396^{(8)}$	$0.706^{(6)}$	$0.7289^{(7)}$	$0.5571^{(4)}$	$0.7861^{(9)}$	$59.8634^{(15)}$	$0.6371^{(5)}$	$0^{(1)}$	$1.1693^{(11)}$	$0.0125^{(2)}$	$0.1568^{(3)}$	$1.7139^{(14)}$	$1.0262^{(10)}$	$1.5003^{(13)}$
	$MRE(\hat{\lambda})$	$3.2383^{(7)}$	$4.1113^{(13)}$	$3.9304^{(12)}$	$3.8623^{(11)}$	$3.5613^{(10)}$	$2.9605^{(5)}$	$3.468^{(8)}$	$4.1226^{(14)}$	$3e-04^{(1)}$	$3.5253^{(9)}$	$0.6553^{(3)}$	$0.5597^{(2)}$	$3.01^{(6)}$	$4.7691^{(15)}$	$2.7174^{(4)}$
	$MRE(\hat{c})$	$2.6617^{(15)}$	$1.6115^{(12)}$	$1.1113^{(4)}$	$1.6581^{(13)}$	$1.4449^{(11)}$	$1.1533^{(5)}$	$1.2877^{(7)}$	$1.943^{(14)}$	$0.0555^{(1)}$	$1.3725^{(9)}$	$0.2057^{(2)}$	$0.3808^{(3)}$	$1.3801^{(10)}$	$1.1609^{(6)}$	$1.2929^{(8)}$
	$MRE(\hat{k})$	$1.3131^{(8)}$	$1.149^{(6)}$	$1.4022^{(10)}$	$0.9917^{(5)}$	$0.9258^{(4)}$	$1.6083^{(12)}$	$1.3739^{(9)}$	$3.8203^{(15)}$	$0.2461^{(1)}$	$1.4582^{(11)}$	$0.444^{(2)}$	$0.7096^{(3)}$	$2.6949^{(14)}$	$1.2909^{(7)}$	$2.6942^{(13)}$
	D_{abs}	$0.0753^{(5)}$	$0.0749^{(3)}$	$0.0746^{(2)}$	$0.0797^{(10)}$	$0.0768^{(6)}$	$0.0814^{(12)}$	$0.0809^{(11)}$	$0.0935^{(13)}$	$0.0706^{(1)}$	$0.0779^{(8)}$	$0.0784^{(9)}$	$0.0768^{(7)}$	$0.1019^{(14)}$	$0.0749^{(4)}$	$0.1055^{(15)}$
	D_{max}	$0.1199^{(9)}$	$0.1135^{(3)}$	$0.1176^{(7)}$	$0.1206^{(10)}$	$0.1173^{(6)}$	$0.1264^{(12)}$	$0.1235^{(11)}$	$0.1447^{(13)}$	$0.0993^{(1)}$	$0.1191^{(8)}$	$0.1157^{(5)}$	$0.1096^{(2)}$	$0.1537^{(14)}$	$0.1137^{(4)}$	$0.1588^{(15)}$
	$\sum$ Ranks	$136^{(11)}$	$122^{(8)}$	$111^{(5)}$	$131^{(9)}$	$100^{(4)}$	$121^{(7)}$	$143^{(13)}$	$168^{(15)}$	$14^{(1)}$	$132^{(10)}$	$41^{(2)}$	$46^{(3)}$	$156^{(14)}$	$117^{(6)}$	$142^{(12)}$
	$\|Bias\|(\hat{\delta})$	$1.2434^{(14)}$	$0.9108^{(12)}$	$0.5462^{(7)}$	$0.6823^{(9)}$	$0.3847^{(5)}$	$0.6276^{(8)}$	$0.3427^{(4)}$	$1.3943^{(15)}$	$0.0113^{(1)}$	$0.5347^{(6)}$	$0.0411^{(2)}$	$0.0653^{(3)}$	$0.8378^{(11)}$	$0.8072^{(10)}$	$0.9393^{(13)}$
	$\|Bias\|(\hat{\lambda})$	$3.9985^{(7)}$	$5.8884^{(14)}$	$7.0592^{(15)}$	$4.0975^{(8)}$	$4.6857^{(12)}$	$4.5997^{(10)}$	$4.5959^{(9)}$	$4.6725^{(11)}$	$0.0038^{(1)}$	$3.2464^{(6)}$	$0.3491^{(3)}$	$0.342^{(2)}$	$2.8172^{(5)}$	$5.8602^{(13)}$	$2.024^{(4)}$
	$\|Bias\|(\hat{c})$	$1.3367^{(15)}$	$0.4326^{(7)}$	$0.487^{(8)}$	$0.6897^{(12)}$	$0.1975^{(6)}$	$0.0034^{(1)}$	$0.1332^{(5)}$	$0.865^{(14)}$	$0.0423^{(3)}$	$0.6871^{(11)}$	$0.0262^{(2)}$	$0.0903^{(4)}$	$0.6214^{(9)}$	$0.7016^{(13)}$	$0.6233^{(10)}$
	$\|Bias\|(\hat{k})$	$0.0231^{(8)}$	$0.0253^{(9)}$	$0.0377^{(12)}$	$0.0161^{(4)}$	$0.0285^{(10)}$	$0.0468^{(13)}$	$0.0337^{(11)}$	$0.0147^{(3)}$	$0.005^{(1)}$	$0.0198^{(7)}$	$0.0108^{(2)}$	$0.0172^{(6)}$	$0.0626^{(15)}$	$0.0165^{(5)}$	$0.0611^{(14)}$
	$MSE(\hat{\delta})$	$2.9362^{(9)}$	$6.142^{(14)}$	$1.3991^{(6)}$	$1.8114^{(7)}$	$0.6762^{(4)}$	$2.2808^{(8)}$	$0.7435^{(5)}$	$13.4359^{(15)}$	$0.0054^{(1)}$	$3.0111^{(10)}$	$0.0618^{(2)}$	$0.3612^{(3)}$	$3.2464^{(11)}$	$5.2768^{(13)}$	$3.5584^{(12)}$
	$MSE(\hat{\lambda})$	$72.4548^{(8)}$	$107.4816^{(14)}$	$127.3963^{(15)}$	$70.9248^{(7)}$	$87.1191^{(10)}$	$89.642^{(12)}$	$88.333^{(11)}$	$80.7699^{(9)}$	$9e-04^{(1)}$	$64.411^{(6)}$	$6.3017^{(2)}$	$7.3737^{(3)}$	$58.0863^{(5)}$	$101.0315^{(13)}$	$45.0099^{(4)}$
	$MSE(\hat{c})$	$5.3894^{(15)}$	$1.5577^{(7)}$	$1.9704^{(8)}$	$2.3733^{(9)}$	$0.768^{(4)}$	$0.4189^{(3)}$	$0.7914^{(5)}$	$3.7783^{(14)}$	$0.009^{(1)}$	$2.4911^{(11)}$	$0.3332^{(2)}$	$0.8848^{(6)}$	$2.9142^{(12)}$	$2.4553^{(10)}$	$2.9305^{(13)}$
20	$MSE(\hat{k})$	$0.0058^{(8)}$	$0.0061^{(10)}$	$0.0096^{(13)}$	$0.0044^{(6)}$	$0.0063^{(11)}$	$0.0087^{(12)}$	$0.0059^{(9)}$	$0.0037^{(4)}$	$5e-04^{(1)}$	$0.0051^{(7)}$	$0.0013^{(2)}$	$0.0022^{(3)}$	$0.0163^{(15)}$	$0.0041^{(5)}$	$0.0157^{(14)}$
	$MRE(\hat{\delta})$	$1.0594^{(13)}$	$0.7738^{(9)}$	$0.4649^{(6)}$	$0.6542^{(8)}$	$0.3606^{(4)}$	$0.6052^{(7)}$	$0.4319^{(5)}$	$1.4029^{(15)}$	$0.0094^{(1)}$	$0.8539^{(10)}$	$0.0441^{(2)}$	$0.2425^{(3)}$	$0.8802^{(11)}$	$1.1059^{(14)}$	$0.8815^{(12)}$
	$MRE(\hat{\lambda})$	$2.2802^{(7)}$	$3.1271^{(13)}$	$3.7305^{(15)}$	$2.3211^{(8)}$	$2.6093^{(10)}$	$2.7009^{(12)}$	$2.6618^{(11)}$	$2.5648^{(9)}$	$0.0022^{(1)}$	$2.1951^{(6)}$	$0.1616^{(2)}$	$0.3147^{(3)}$	$2.0547^{(5)}$	$3.1449^{(14)}$	$1.6671^{(4)}$
	$MRE(\hat{c})$	$2.3533^{(15)}$	$1.0566^{(7)}$	$1.1898^{(8)}$	$1.3785^{(9)}$	$0.7248^{(5)}$	$0.6661^{(4)}$	$0.7805^{(6)}$	$1.6302^{(14)}$	$0.0582^{(1)}$	$1.4354^{(10)}$	$0.2515^{(2)}$	$0.5908^{(3)}$	$1.5386^{(13)}$	$1.4449^{(11)}$	$1.5318^{(12)}$
	$MRE(\hat{k})$	$1.3002^{(13)}$	$1.0068^{(8)}$	$1.2709^{(12)}$	$0.9503^{(5)}$	$0.9915^{(7)}$	$1.2531^{(11)}$	$1.0388^{(9)}$	$0.8921^{(4)}$	$0.2175^{(1)}$	$1.0526^{(10)}$	$0.4051^{(2)}$	$0.5733^{(3)}$	$1.8961^{(15)}$	$0.9629^{(6)}$	$1.8715^{(14)}$
	D_{abs}	$0.0532^{(2)}$	$0.0547^{(4)}$	$0.0565^{(7)}$	$0.0594^{(10)}$	$0.0569^{(9)}$	$0.0601^{(12)}$	$0.0566^{(8)}$	$0.0669^{(13)}$	$0.0505^{(1)}$	$0.0551^{(5)}$	$0.0597^{(11)}$	$0.054^{(3)}$	$0.0723^{(14)}$	$0.0555^{(6)}$	$0.0729^{(15)}$
	D_{max}	$0.0858^{(5)}$	$0.0836^{(3)}$	$0.0872^{(7)}$	$0.0907^{(11)}$	$0.085^{(4)}$	$0.0933^{(12)}$	$0.0872^{(8)}$	$0.1028^{(13)}$	$0.0736^{(1)}$	$0.0886^{(9)}$	$0.0889^{(10)}$	$0.0818^{(2)}$	$0.1154^{(14)}$	$0.0871^{(6)}$	$0.1181^{(15)}$
	$\sum$ Ranks	$139^{(11)}$	$131^{(9)}$	$139^{(11)}$	$113^{(6)}$	$101^{(4)}$	$125^{(8)}$	$106^{(5)}$	$153^{(13)}$	$16^{(1)}$	$114^{(7)}$	$46^{(2)}$	$47^{(3)}$	$155^{(14)}$	$139^{(11)}$	$156^{(15)}$
	$\|Bias\|(\hat{\delta})$	$0.6994^{(15)}$	$0.1781^{(10)}$	$0.0178^{(3)}$	$0.0792^{(7)}$	$0.0787^{(6)}$	$0.1173^{(9)}$	$0.0045^{(1)}$	$0.5156^{(14)}$	$0.0134^{(2)}$	$0.2091^{(11)}$	$0.0639^{(5)}$	$0.1025^{(8)}$	$0.0528^{(4)}$	$0.4446^{(13)}$	$0.2098^{(12)}$
	$\|Bias\|(\hat{\lambda})$	$3.9436^{(14)}$	$1.8621^{(6)}$	$3.1953^{(12)}$	$2.0849^{(7)}$	$3.4676^{(13)}$	$2.2165^{(8)}$	$2.7331^{(10)}$	$2.9782^{(11)}$	$0.0128^{(2)}$	$2.3044^{(9)}$	$0.0017^{(1)}$	$0.0421^{(3)}$	$1.2046^{(4)}$	$4.1698^{(15)}$	$1.484^{(5)}$
	$\|Bias\|(\hat{c})$	$0.315^{(12)}$	$0.2197^{(9)}$	$0.1093^{(7)}$	$0.2776^{(11)}$	$0.0244^{(6)}$	$0.0048^{(2)}$	$1e-04^{(1)}$	$0.3636^{(14)}$	$0.0172^{(4)}$	$0.3321^{(13)}$	$0.0162^{(3)}$	$0.0224^{(5)}$	$0.2709^{(10)}$	$0.4358^{(15)}$	$0.1823^{(8)}$
	$\|Bias\|(\hat{k})$	$0.0326^{(15)}$	$0.0037^{(4)}$	$0.0145^{(10)}$	$5e-04^{(1)}$	$0.0147^{(11)}$	$0.0165^{(12)}$	$0.0101^{(7)}$	$0.0107^{(8)}$	$0.0029^{(2)}$	$0.0118^{(9)}$	$0.0036^{(3)}$	$0.0057^{(6)}$	$0.0222^{(13)}$	$0.0046^{(5)}$	$0.0276^{(14)}$
	$MSE(\hat{\delta})$	$1.125^{(11)}$	$0.2079^{(7)}$	$0.1342^{(4)}$	$0.3038^{(8)}$	$0.1437^{(5)}$	$0.2073^{(6)}$	$0.3114^{(9)}$	$4.6485^{(14)}$	$0.0088^{(1)}$	$1.2567^{(12)}$	$0.0533^{(2)}$	$0.0718^{(3)}$	$0.6395^{(10)}$	$4.828^{(15)}$	$1.4044^{(13)}$
	$MSE(\hat{\lambda})$	$54.5663^{(12)}$	$34.2657^{(4)}$	$61.0201^{(13)}$	$40.3517^{(7)}$	$65.0973^{(14)}$	$47.3132^{(9)}$	$51.0183^{(11)}$	$49.6116^{(10)}$	$0.0083^{(2)}$	$40.0778^{(6)}$	$7e-04^{(1)}$	$0.0378^{(3)}$	$34.8264^{(5)}$	$72.5371^{(15)}$	$40.9113^{(8)}$
	$MSE(\hat{c})$	$1.6983^{(15)}$	$0.3451^{(7)}$	$0.3692^{(8)}$	$0.4211^{(9)}$	$0.0753^{(4)}$	$0.1481^{(6)}$	$0.0766^{(5)}$	$0.8095^{(11)}$	$0.001^{(1)}$	$0.8544^{(13)}$	$0.0293^{(2)}$	$0.0572^{(3)}$	$0.8286^{(12)}$	$0.9867^{(14)}$	$0.6922^{(10)}$
50	$MSE(\hat{k})$	$0.003^{(13)}$	$0.0013^{(6)}$	$0.0025^{(12)}$	$0.001^{(5)}$	$0.0017^{(8)}$	$0.0021^{(10)}$	$0.001^{(4)}$	$0.002^{(9)}$	$1e-04^{(1)}$	$0.0022^{(11)}$	$3e-04^{(2)}$	$4e-04^{(3)}$	$0.0041^{(14)}$	$0.0015^{(7)}$	$0.0043^{(15)}$
	$MRE(\hat{\delta})$	$0.6034^{(12)}$	$0.2558^{(7)}$	$0.1604^{(4)}$	$0.3305^{(8)}$	$0.1941^{(5)}$	$0.2333^{(6)}$	$0.3438^{(9)}$	$0.6946^{(14)}$	$0.0112^{(1)}$	$0.6484^{(13)}$	$0.0759^{(2)}$	$0.1477^{(3)}$	$0.4629^{(10)}$	$0.8974^{(15)}$	$0.5938^{(11)}$
	$MRE(\hat{\lambda})$	$1.9013^{(12)}$	$1.3399^{(5)}$	$1.9179^{(13)}$	$1.3827^{(6)}$	$2.0623^{(14)}$	$1.6075^{(8)}$	$1.8266^{(11)}$	$1.6773^{(10)}$	$0.0059^{(2)}$	$1.6561^{(9)}$	$0.0056^{(1)}$	$0.0328^{(3)}$	$1.3277^{(4)}$	$2.3283^{(15)}$	$1.4612^{(7)}$
	$MRE(\hat{c})$	$1.1456^{(15)}$	$0.5394^{(8)}$	$0.4442^{(7)}$	$0.5809^{(9)}$	$0.2901^{(4)}$	$0.3485^{(6)}$	$0.293^{(5)}$	$0.8424^{(12)}$	$0.0244^{(1)}$	$0.8658^{(13)}$	$0.1177^{(2)}$	$0.2172^{(3)}$	$0.8011^{(11)}$	$0.9415^{(14)}$	$0.759^{(10)}$
	$MRE(\hat{k})$	$1.0022^{(15)}$	$0.5049^{(6)}$	$0.5676^{(8)}$	$0.4477^{(5)}$	$0.5156^{(7)}$	$0.5738^{(9)}$	$0.44^{(4)}$	$0.7241^{(11)}$	$0.1207^{(1)}$	$0.7529^{(12)}$	$0.2253^{(2)}$	$0.2739^{(3)}$	$0.9252^{(13)}$	$0.6585^{(10)}$	$0.9748^{(14)}$
	D_{abs}	$0.0327^{(1)}$	$0.0341^{(3)}$	$0.0363^{(8)}$	$0.0361^{(7)}$	$0.0366^{(10)}$	$0.0375^{(12)}$	$0.0358^{(5)}$	$0.0401^{(13)}$	$0.034^{(2)}$	$0.0345^{(4)}$	$0.0373^{(11)}$	$0.0364^{(9)}$	$0.0473^{(15)}$	$0.0359^{(6)}$	$0.0472^{(14)}$
	D_{max}	$0.0524^{(2)}$	$0.0526^{(3)}$	$0.0595^{(12)}$	$0.0539^{(6)}$	$0.0581^{(10)}$	$0.0592^{(11)}$	$0.0549^{(7)}$	$0.0602^{(13)}$	$0.0503^{(1)}$	$0.0539^{(5)}$	$0.0554^{(8)}$	$0.0537^{(4)}$	$0.0768^{(14)}$	$0.0562^{(9)}$	$0.0771^{(15)}$
	$\sum$ Ranks	$164^{(14)}$	$85^{(4)}$	$121^{(9)}$	$96^{(6)}$	$117^{(8)}$	$114^{(7)}$	$89^{(5)}$	$164^{(14)}$	$23^{(1)}$	$140^{(11)}$	$45^{(2)}$	$59^{(3)}$	$139^{(10)}$	$168^{(15)}$	$156^{(12)}$
	$\|Bias\|(\hat{\delta})$	$0.2971^{(15)}$	$0.1084^{(13)}$	$0.0338^{(5)}$	$0.1024^{(11)}$	$0.012^{(3)}$	$0.0615^{(7)}$	$0.0746^{(9)}$	$0.0027^{(2)}$	$0^{(1)}$	$0.1069^{(12)}$	$0.0405^{(6)}$	$0.024^{(4)}$	$0.0717^{(8)}$	$0.1346^{(14)}$	$0.0883^{(10)}$
	$\|Bias\|(\hat{\lambda})$	$4.1106^{(15)}$	$2.1406^{(8)}$	$2.304^{(10)}$	$2.0475^{(7)}$	$1.2548^{(4)}$	$1.2818^{(5)}$	$3.3677^{(14)}$	$2.3611^{(11)}$	$3e-04^{(1)}$	$2.364^{(12)}$	$0.0032^{(2)}$	$0.3286^{(3)}$	$2.2794^{(9)}$	$3.0108^{(13)}$	$1.8018^{(6)}$
	$\|Bias\|(\hat{c})$	$0.1273^{(12)}$	$0.0599^{(9)}$	$0.042^{(7)}$	$0.0985^{(11)}$	$0.0634^{(10)}$	$0.0131^{(2)}$	$0.0277^{(4)}$	$0.1582^{(14)}$	$0.0125^{(1)}$	$0.1294^{(13)}$	$0.0198^{(3)}$	$0.0468^{(8)}$	$0.0401^{(6)}$	$0.3832^{(15)}$	$0.037^{(5)}$
	$\|Bias\|(\hat{k})$	$0.029^{(14)}$	$0.0071^{(5)}$	$0.012^{(11)}$	$0.0059^{(4)}$	$0.0072^{(6)}$	$0.0104^{(9)}$	$0.0147^{(12)}$	$0.0097^{(8)}$	$0.0013^{(1)}$	$0.0116^{(10)}$	$0.0042^{(3)}$	$0.0094^{(7)}$	$0.029^{(15)}$	$0.0022^{(2)}$	$0.0276^{(13)}$
	$MSE(\hat{\delta})$	$0.2747^{(10)}$	$0.2581^{(9)}$	$0.025^{(2)}$	$0.5198^{(14)}$	$0.0493^{(6)}$	$0.0354^{(4)}$	$0.1811^{(7)}$	$0.2459^{(8)}$	$0^{(1)}$	$0.4926^{(13)}$	$0.0275^{(3)}$	$0.0404^{(5)}$	$0.3133^{(11)}$	$0.5554^{(15)}$	$0.3656^{(12)}$
	$MSE(\hat{\lambda})$	$52.2789^{(15)}$	$27.5721^{(8)}$	$32.8477^{(11)}$	$26.2305^{(7)}$	$11.8879^{(4)}$	$19.7884^{(5)}$	$45.1132^{(14)}$	$25.3938^{(6)}$	$0^{(1)}$	$30.0721^{(9)}$	$2e-04^{(2)}$	$6.5125^{(3)}$	$36.3958^{(12)}$	$42.4948^{(13)}$	$31.369^{(10)}$
	$MSE(\hat{c})$	$0.4507^{(14)}$	$0.1158^{(5)}$	$0.1808^{(8)}$	$0.1824^{(9)}$	$0.1705^{(7)}$	$0.1017^{(3)}$	$0.1039^{(4)}$	$0.4377^{(13)}$	$4e-04^{(1)}$	$0.4307^{(12)}$	$0.0367^{(2)}$	$0.146^{(6)}$	$0.3236^{(11)}$	$0.8849^{(15)}$	$0.3205^{(10)}$
100	$MSE(\hat{k})$	$0.0013^{(12)}$	$8e-04^{(6)}$	$0.0014^{(13)}$	$7e-04^{(4)}$	$7e-04^{(5)}$	$9e-04^{(7)}$	$0.0011^{(10)}$	$0.001^{(8)}$	$0^{(1)}$	$0.0012^{(11)}$	$2e-04^{(2)}$	$6e-04^{(3)}$	$0.0026^{(15)}$	$0.001^{(9)}$	$0.0024^{(14)}$
	$MRE(\hat{\delta})$	$0.2476^{(8)}$	$0.2606^{(9)}$	$0.0642^{(2)}$	$0.3499^{(13)}$	$0.0794^{(5)}$	$0.0752^{(3)}$	$0.2353^{(7)}$	$0.2748^{(10)}$	$0^{(1)}$	$0.4012^{(14)}$	$0.0781^{(4)}$	$0.1222^{(6)}$	$0.2994^{(11)}$	$0.4226^{(15)}$	$0.3299^{(12)}$
	$MRE(\hat{\lambda})$	$1.8685^{(15)}$	$1.1508^{(7)}$	$1.3447^{(11)}$	$1.0806^{(6)}$	$0.6991^{(4)}$	$0.7897^{(5)}$	$1.7469^{(14)}$	$1.1766^{(8)}$	$1e-04^{(1)}$	$1.2493^{(10)}$	$0.0034^{(2)}$	$0.1992^{(3)}$	$1.401^{(12)}$	$1.6491^{(13)}$	$1.1932^{(9)}$
	$MRE(\hat{c})$	$0.4993^{(10)}$	$0.3647^{(7)}$	$0.3772^{(8)}$	$0.3996^{(9)}$	$0.3434^{(6)}$	$0.2574^{(4)}$	$0.3356^{(5)}$	$0.5758^{(13)}$	$0.0173^{(1)}$	$0.5917^{(14)}$	$0.1492^{(2)}$	$0.2546^{(3)}$	$0.5153^{(11)}$	$0.8505^{(15)}$	$0.5215^{(12)}$
	$MRE(\hat{k})$	$0.6444^{(13)}$	$0.4108^{(6)}$	$0.4799^{(8)}$	$0.4185^{(7)}$	$0.389^{(5)}$	$0.3637^{(4)}$	$0.4833^{(9)}$	$0.5275^{(10)}$	$0.0695^{(1)}$	$0.5618^{(12)}$	$0.2045^{(2)}$	$0.2962^{(3)}$	$0.7912^{(15)}$	$0.5482^{(11)}$	$0.7674^{(14)}$
	D_{abs}	$0.0217^{(2)}$	$0.0227^{(4)}$	$0.023^{(5)}$	$0.024^{(10)}$	$0.0232^{(6)}$	$0.0232^{(7)}$	$0.0233^{(8)}$	$0.0247^{(11)}$	$0.0209^{(1)}$	$0.0219^{(3)}$	$0.0281^{(15)}$	$0.0235^{(9)}$	$0.028^{(13)}$	$0.025^{(12)}$	$0.0281^{(14)}$
	D_{max}	$0.0369^{(7)}$	$0.0361^{(3)}$	$0.0377^{(8)}$	$0.0378^{(9)}$	$0.0362^{(4)}$	$0.0362^{(5)}$	$0.0382^{(10)}$	$0.0399^{(11)}$	$0.0308^{(1)}$	$0.0359^{(2)}$	$0.0414^{(13)}$	$0.0368^{(6)}$	$0.0469^{(14)}$	$0.0411^{(12)}$	$0.047^{(15)}$
	$\sum$ Ranks	$162^{(13)}$	$99^{(6)}$	$109^{(7)}$	$121^{(8)}$	$75^{(5)}$	$70^{(4)}$	$127^{(9)}$	$133^{(10)}$	$14^{(1)}$	$147^{(11)}$	$61^{(2)}$	$69^{(3)}$	$163^{(14)}$	$174^{(15)}$	$156^{(12)}$
	$\|Bias\|(\hat{\delta})$	$0.13^{(12)}$	$0.0342^{(6)}$	$0.0514^{(10)}$	$0.0512^{(9)}$	$0.0427^{(8)}$	$0.0402^{(7)}$	$0.0052^{(3)}$	$0.001^{(2)}$	$0^{(1)}$	$0.0751^{(11)}$	$0.0208^{(5)}$	$0.0144^{(4)}$	$0.1645^{(15)}$	$0.1325^{(13)}$	$0.1383^{(14)}$
	$\|Bias\|(\hat{\lambda})$	$1.5422^{(13)}$	$1.35^{(11)}$	$2.7712^{(15)}$	$0.2948^{(4)}$	$1.4338^{(12)}$	$0.8758^{(9)}$	$1.6328^{(14)}$	$0.7955^{(7)}$	$2e-04^{(1)}$	$0.4623^{(5)}$	$0.0012^{(2)}$	$0.0326^{(3)}$	$0.9613^{(10)}$	$0.8523^{(8)}$	$0.7234^{(6)}$
	$\|Bias\|(\hat{c})$	$0.1057^{(12)}$	$0.0043^{(1)}$	$0.094^{(11)}$	$0.0341^{(6)}$	$0.0488^{(9)}$	$0.0575^{(10)}$	$0.036^{(8)}$	$0.1274^{(13)}$	$0.0089^{(2)}$	$0.1664^{(14)}$	$0.0192^{(5)}$	$0.0345^{(7)}$	$0.0179^{(4)}$	$0.4661^{(15)}$	$0.0133^{(3)}$
	$\|Bias\|(\hat{k})$	$0.0131^{(14)}$	$0.0074^{(8)}$	$0.0192^{(15)}$	$0.0011^{(5)}$	$0.0113^{(13)}$	$0.011^{(12)}$	$0.0086^{(9)}$	$0^{(1)}$	$7e-04^{(4)}$	$2e-04^{(2)}$	$0.002^{(6)}$	$6e-04^{(3)}$	$0.011^{(11)}$	$0.0054^{(7)}$	$0.0099^{(10)}$
	$MSE(\hat{\delta})$	$0.0551^{(8)}$	$0.0874^{(11)}$	$0.0173^{(5)}$	$0.0838^{(10)}$	$0.0158^{(4)}$	$0.0127^{(3)}$	$0.0229^{(6)}$	$0.0667^{(9)}$	$0^{(1)}$	$0.128^{(12)}$	$0.0047^{(2)}$	$0.027^{(7)}$	$0.1684^{(15)}$	$0.158^{(14)}$	$0.1405^{(13)}$
	$MSE(\hat{\lambda})$	$15.9459^{(10)}$	$16.1157^{(11)}$	$44.373^{(15)}$	$2.1279^{(4)}$	$17.0324^{(12)}$	$8.1016^{(6)}$	$21.4782^{(14)}$	$9.8759^{(7)}$	$0^{(1)}$	$4.8454^{(5)}$	$3e-04^{(2)}$	$0.0192^{(3)}$	$20.917^{(13)}$	$10.9246^{(8)}$	$14.6463^{(9)}$
	$MSE(\hat{c})$	$0.0794^{(10)}$	$0.0733^{(8)}$	$0.0765^{(9)}$	$0.0441^{(4)}$	$0.0612^{(7)}$	$0.0566^{(6)}$	$0.0447^{(5)}$	$0.1398^{(12)}$	$2e-04^{(1)}$	$0.2481^{(14)}$	$0.0129^{(2)}$	$0.0306^{(3)}$	$0.1399^{(13)}$	$0.9194^{(15)}$	$0.1371^{(11)}$
200	$MSE(\hat{k})$	$3e-04^{(5)}$	$5e-04^{(8)}$	$0.0015^{(15)}$	$2e-04^{(4)}$	$8e-04^{(13)}$	$7e-04^{(11)}$	$5e-04^{(9)}$	$4e-04^{(6)}$	$0^{(1)}$	$4e-04^{(7)}$	$1e-04^{(2)}$	$1e-04^{(3)}$	$9e-04^{(14)}$	$6e-04^{(10)}$	$7e-04^{(12)}$
	$MRE(\hat{\delta})$	$0.1126^{(8)}$	$0.1325^{(11)}$	$0.0479^{(5)}$	$0.1167^{(9)}$	$0.0396^{(3)}$	$0.0383^{(2)}$	$0.0764^{(6)}$	$0.1306^{(10)}$	$0^{(1)}$	$0.1881^{(13)}$	$0.0413^{(4)}$	$0.0806^{(7)}$	$0.1994^{(14)}$	$0.2157^{(15)}$	$0.1747^{(12)}$
	$MRE(\hat{\lambda})$	$0.7373^{(9)}$	$0.7447^{(10)}$	$1.4561^{(15)}$	$0.2949^{(4)}$	$0.7683^{(11)}$	$0.532^{(6)}$	$0.9821^{(14)}$	$0.6577^{(8)}$	$1e-04^{(1)}$	$0.5272^{(5)}$	$0.0026^{(2)}$	$0.0286^{(3)}$	$0.7988^{(13)}$	$0.7957^{(12)}$	$0.5943^{(7)}$
	$MRE(\hat{c})$	$0.23^{(8)}$	$0.2656^{(9)}$	$0.2737^{(10)}$	$0.2141^{(4)}$	$0.2295^{(7)}$	$0.2214^{(6)}$	$0.2

Table 2: Partial and overall ranks for all estimation methods of the SHBXII model from Table 1.

n	Est.	MLE	ADE	RTADE	LTADE	CVME	OLSE	WLSE	ADSOE	KE	MPSE	MSADE	MSALDE	MSSDE	MSSLDE	MSLNDE
	$\lvert Bias\rvert(\hat{\delta})$	13	10	7	8	5	9	15	4	1	11	2	3	14	6	12
	$\lvert Bias\rvert(\hat{\lambda})$	7	13	11	12	10	6	8	14	1	9	3	2	5	15	4
	$\lvert Bias\rvert(\hat{c})$	15	12	7	13	11	5	9	14	1	10	2	3	6	8	4
	$\lvert Bias\rvert(\hat{k})$	3	6	11	4	5	12	10	15	1	9	2	7	13	8	14
	$MSE(\hat{\delta})$	12	6	7	9	5	10	15	4	1	11	2	3	14	8	13
	$MSE(\hat{\lambda})$	7	13	11	12	10	6	8	14	1	9	3	2	5	15	4
	$MSE(\hat{c})$	15	11	5	12	9	6	8	14	1	7	2	3	13	4	10
10	$MSE(\hat{k})$	8	6	11	5	4	12	9	15	1	10	2	3	14	7	13
	$MRE(\hat{\delta})$	12	8	6	7	4	9	15	5	1	11	2	3	14	10	13
	$MRE(\hat{\lambda})$	7	13	12	11	10	5	8	14	1	9	3	2	6	15	4
	$MRE(\hat{c})$	15	12	4	13	11	5	7	14	1	9	2	3	10	6	8
	$MRE(\hat{k})$	8	6	10	5	4	12	9	15	1	11	2	3	14	7	13
	D_{abs}	5	3	2	10	6	12	11	13	1	8	9	7	14	4	15
	D_{max}	9	3	7	10	6	12	11	13	1	8	5	2	14	4	15
	$\sum$ Ranks	$136^{\{11\}}$	$122^{\{8\}}$	$111^{\{5\}}$	$131^{\{9\}}$	$100^{\{4\}}$	$121^{\{7\}}$	$143^{\{13\}}$	$168^{\{15\}}$	$14^{\{1\}}$	$132^{\{10\}}$	$41^{\{2\}}$	$46^{\{3\}}$	$156^{\{14\}}$	$117^{\{6\}}$	$142^{\{12\}}$
	$\lvert Bias\rvert(\hat{\delta})$	14	12	7	9	5	8	4	15	1	6	2	3	11	10	13
	$\lvert Bias\rvert(\hat{\lambda})$	7	14	15	8	12	10	9	11	1	6	3	2	5	13	4
	$\lvert Bias\rvert(\hat{c})$	15	7	8	12	6	1	5	14	3	11	2	4	9	13	10
	$\lvert Bias\rvert(\hat{k})$	8	9	12	4	10	13	11	3	1	7	2	6	15	5	14
	$MSE(\hat{\delta})$	9	14	6	7	4	8	5	15	1	10	2	3	11	13	12
	$MSE(\hat{\lambda})$	8	14	15	7	10	12	11	9	1	6	2	3	5	13	4
	$MSE(\hat{c})$	15	7	8	9	4	3	5	14	1	11	2	6	12	10	13
20	$MSE(\hat{k})$	8	10	13	6	11	12	9	4	1	7	2	3	15	5	14
	$MRE(\hat{\delta})$	13	9	6	8	4	7	5	15	1	10	2	3	11	14	12
	$MRE(\hat{\lambda})$	7	13	15	8	10	12	11	9	1	6	2	3	5	14	4
	$MRE(\hat{c})$	15	7	8	9	5	4	6	14	1	10	2	3	13	11	12
	$MRE(\hat{k})$	13	8	12	5	7	11	9	4	1	10	2	3	15	6	14
	D_{abs}	2	4	7	10	9	12	8	13	1	5	11	3	14	6	15
	D_{max}	5	3	7	11	4	12	8	13	1	9	10	2	14	6	15
	$\sum$ Ranks	$139^{\{11\}}$	$131^{\{9\}}$	$139^{\{11\}}$	$113^{\{6\}}$	$101^{\{4\}}$	$125^{\{8\}}$	$106^{\{5\}}$	$153^{\{13\}}$	$16^{\{1\}}$	$114^{\{7\}}$	$46^{\{2\}}$	$47^{\{3\}}$	$155^{\{14\}}$	$139^{\{11\}}$	$156^{\{15\}}$
	$\lvert Bias\rvert(\hat{\delta})$	15	10	3	7	6	9	1	14	2	11	5	8	4	13	12
	$\lvert Bias\rvert(\hat{\lambda})$	14	6	12	7	13	8	10	11	2	9	1	3	4	15	5
	$\lvert Bias\rvert(\hat{c})$	12	9	7	11	6	2	1	14	4	13	3	5	10	15	8
	$\lvert Bias\rvert(\hat{k})$	15	4	10	1	11	12	7	8	2	9	3	6	13	5	14
	$MSE(\hat{\delta})$	11	7	4	8	5	6	9	14	1	12	2	3	10	15	13
	$MSE(\hat{\lambda})$	12	4	13	7	14	9	11	10	2	6	1	3	5	15	8
	$MSE(\hat{c})$	15	7	8	9	4	6	5	11	1	13	2	3	12	14	10
50	$MSE(\hat{k})$	13	6	12	5	8	10	4	9	1	11	2	3	14	7	15
	$MRE(\hat{\delta})$	12	7	4	8	5	6	9	14	1	13	2	3	10	15	11
	$MRE(\hat{\lambda})$	12	5	13	6	14	8	11	10	2	9	1	3	4	15	7
	$MRE(\hat{c})$	15	8	7	9	4	6	5	12	1	13	2	3	11	14	10
	$MRE(\hat{k})$	15	6	8	5	7	9	4	11	1	12	2	3	13	10	14
	D_{abs}	1	3	8	7	10	12	5	13	2	4	11	9	15	6	14
	D_{max}	2	3	12	6	10	11	7	13	1	5	8	4	14	9	15
	$\sum$ Ranks	$164^{\{14\}}$	$85^{\{4\}}$	$121^{\{9\}}$	$96^{\{6\}}$	$117^{\{8\}}$	$114^{\{7\}}$	$89^{\{5\}}$	$164^{\{14\}}$	$23^{\{1\}}$	$140^{\{11\}}$	$45^{\{2\}}$	$59^{\{3\}}$	$139^{\{10\}}$	$168^{\{15\}}$	$156^{\{12\}}$
	$\lvert Bias\rvert(\hat{\delta})$	15	13	5	11	3	7	9	2	1	12	6	4	8	14	10
	$\lvert Bias\rvert(\hat{\lambda})$	15	8	10	7	4	5	14	11	1	12	2	3	9	13	6
	$\lvert Bias\rvert(\hat{c})$	12	9	7	11	10	2	4	14	1	13	3	8	6	15	5
	$\lvert Bias\rvert(\hat{k})$	14	5	11	4	6	9	12	8	1	10	3	7	15	2	13
	$MSE(\hat{\delta})$	10	9	2	14	6	4	7	8	1	13	3	5	11	15	12
	$MSE(\hat{\lambda})$	15	8	11	7	4	5	14	6	1	9	2	3	12	13	10
	$MSE(\hat{c})$	14	5	8	9	7	3	4	13	1	12	2	6	11	15	10
100	$MSE(\hat{k})$	12	6	13	4	5	7	10	8	1	11	2	3	15	9	14
	$MRE(\hat{\delta})$	8	9	2	13	5	3	7	10	1	14	4	6	11	15	12
	$MRE(\hat{\lambda})$	15	7	11	6	4	5	14	8	1	10	2	3	12	13	9
	$MRE(\hat{c})$	10	7	8	9	6	4	5	13	1	14	2	3	11	15	12
	$MRE(\hat{k})$	13	6	8	7	5	4	9	10	1	12	2	3	15	11	14
	D_{abs}	2	4	5	10	6	7	8	11	1	3	15	9	13	12	14
	D_{max}	7	3	8	9	4	5	10	11	1	2	13	6	14	12	15
	$\sum$ Ranks	$162^{\{13\}}$	$99^{\{6\}}$	$109^{\{7\}}$	$121^{\{8\}}$	$75^{\{5\}}$	$70^{\{4\}}$	$127^{\{9\}}$	$133^{\{10\}}$	$14^{\{1\}}$	$147^{\{11\}}$	$61^{\{2\}}$	$69^{\{3\}}$	$163^{\{14\}}$	$174^{\{15\}}$	$156^{\{12\}}$
	$\lvert Bias\rvert(\hat{\delta})$	12	6	10	9	8	7	3	2	1	11	5	4	15	13	14
	$\lvert Bias\rvert(\hat{\lambda})$	13	11	15	4	12	9	14	7	1	5	2	3	10	8	6
	$\lvert Bias\rvert(\hat{c})$	12	1	11	6	9	10	8	13	2	14	5	7	4	15	3
	$\lvert Bias\rvert(\hat{k})$	14	8	15	5	13	12	9	1	4	2	6	3	11	7	10
	$MSE(\hat{\delta})$	8	11	5	10	4	3	6	9	1	12	2	7	15	14	13
	$MSE(\hat{\lambda})$	10	11	15	4	12	6	14	7	1	5	2	3	13	8	9
	$MSE(\hat{c})$	10	8	9	4	7	6	5	12	1	14	2	3	13	15	11
200	$MSE(\hat{k})$	5	8	15	4	13	11	9	6	1	7	2	3	14	10	12
	$MRE(\hat{\delta})$	8	11	5	9	3	2	6	10	1	13	4	7	14	15	12
	$MRE(\hat{\lambda})$	9	10	15	4	11	6	14	8	1	5	2	3	13	12	7
	$MRE(\hat{c})$	8	9	10	4	7	6	5	12	1	14	2	3	13	15	11
	$MRE(\hat{k})$	5	7	15	4	10	9	6	8	1	11	2	3	14	12	13
	D_{abs}	3	2	10	5	9	7	6	8	1	4	13	11	14	12	15
	D_{max}	2	5	13	3	11	7	8	10	1	4	9	6	15	12	14
	$\sum$ Ranks	$119^{\{9\}}$	$108^{\{6\}}$	$163^{\{13\}}$	$75^{\{4\}}$	$129^{\{11\}}$	$101^{\{5\}}$	$113^{\{8\}}$	$113^{\{8\}}$	$18^{\{1\}}$	$121^{\{10\}}$	$58^{\{2\}}$	$66^{\{3\}}$	$178^{\{15\}}$	$168^{\{14\}}$	$150^{\{12\}}$
	$\lvert Bias\rvert(\hat{\delta})$	15	14	9	11	6	7	8	5	1	4	2	3	10	12	13
	$\lvert Bias\rvert(\hat{\lambda})$	15	7	9	8	4	6	12	14	1	11	2	3	5	13	10
	$\lvert Bias\rvert(\hat{c})$	5	13	8	14	11	7	9	10	2	12	1	3	4	15	6
	$\lvert Bias\rvert(\hat{k})$	15	11	10	12	5	7	2	9	3	1	6	4	13	8	14
	$MSE(\hat{\delta})$	10	15	5	12	3	4	7	9	1	8	2	6	14	11	13
	$MSE(\hat{\lambda})$	13	5	11	8	4	6	12	10	1	9	2	3	7	14	15
	$MSE(\hat{c})$	7	5	13	6	11	8	4	9	1	12	2	3	10	15	14
300	$MSE(\hat{k})$	7	5	12	4	10	8	6	9	1	11	2	3	14	13	15
	$MRE(\hat{\delta})$	8	15	5	13	3	4	7	9	1	10	2	6	14	11	12
	$MRE(\hat{\lambda})$	15	7	10	8	4	5	12	13	1	11	2	3	6	14	9
	$MRE(\hat{c})$	4	10	7	9	8	5	6	11	1	12	2	3	13	15	14
	$MRE(\hat{k})$	9	8	7	4	10	5	6	11	1	12	2	3	13	14	15
	D_{abs}	1	4	7	9	10	11	5	13	3	2	12	6	15	8	14
	D_{max}	3	5	12	6	9	11	7	13	1	4	8	2	15	10	14
	$\sum$ Ranks	$127^{\{11\}}$	$124^{\{8\}}$	$125^{\{10\}}$	$124^{\{8\}}$	$98^{\{5\}}$	$94^{\{4\}}$	$103^{\{6\}}$	$145^{\{12\}}$	$19^{\{1\}}$	$119^{\{7\}}$	$47^{\{2\}}$	$51^{\{3\}}$	$153^{\{13\}}$	$173^{\{14\}}$	$178^{\{15\}}$

Table 3: Results from simulation Case II.

n	Est.	MLE	ADE	RTADE	LTADE	CVME	OLSE	WLSE	ADSOE	KE	MPSE	MSADE	MSALDE	MSSDE	MSSLDE	MSLNDE
	$\lvert Bias\rvert(\hat{\delta})$	$0.0492^{(6)}$	$0.0953^{(15)}$	$0.0531^{(7)}$	$0.0906^{(13)}$	$0.025^{(3)}$	$0.085^{(10)}$	$0.0916^{(14)}$	$0.0086^{(1)}$	$0.0237^{(2)}$	$0.0805^{(9)}$	$0.0257^{(4)}$	$0.027^{(5)}$	$0.0859^{(11)}$	$0.0859^{(12)}$	$0.0798^{(8)}$
	$\lvert Bias\rvert(\hat{\lambda})$	$5.0091^{(15)}$	$1.581^{(5)}$	$1.245^{(4)}$	$2.1008^{(9)}$	$1.5825^{(6)}$	$1.7551^{(7)}$	$1.9005^{(8)}$	$2.6176^{(10)}$	$0.0365^{(2)}$	$2.8983^{(12)}$	$0.0657^{(3)}$	$0.0251^{(1)}$	$3.0801^{(14)}$	$2.7232^{(11)}$	$2.9636^{(13)}$
	$\lvert Bias\rvert(\hat{c})$	$0.0081^{(1)}$	$1.1713^{(12)}$	$1.772^{(15)}$	$1.2393^{(14)}$	$1.0836^{(10)}$	$1.1786^{(13)}$	$1.1382^{(11)}$	$0.1587^{(5)}$	$0.0347^{(3)}$	$0.634^{(8)}$	$0.0105^{(2)}$	$0.1381^{(4)}$	$0.6328^{(6)}$	$0.9946^{(9)}$	$0.6339^{(7)}$
	$\lvert Bias\rvert(\hat{k})$	$15.9756^{(15)}$	$9.7468^{(10)}$	$10.3612^{(12)}$	$10.5849^{(13)}$	$10.0198^{(11)}$	$8.1803^{(5)}$	$9.0709^{(6)}$	$7.8648^{(4)}$	$0.0055^{(2)}$	$10.6114^{(14)}$	$9e-04^{(1)}$	$0.0435^{(3)}$	$9.0954^{(7)}$	$9.4569^{(9)}$	$9.4493^{(8)}$
	$MSE(\hat{\delta})$	$0.0526^{(5)}$	$0.0654^{(13)}$	$0.0513^{(4)}$	$0.0655^{(14)}$	$0.0572^{(6)}$	$0.0604^{(9)}$	$0.0622^{(11)}$	$0.0573^{(7)}$	$0.0105^{(1)}$	$0.0633^{(12)}$	$0.017^{(2)}$	$0.027^{(3)}$	$0.0606^{(10)}$	$0.0666^{(15)}$	$0.0587^{(8)}$
	$MSE(\hat{\lambda})$	$53.8929^{(15)}$	$7.9889^{(5)}$	$4.8186^{(4)}$	$17.1497^{(9)}$	$8.6543^{(6)}$	$13.1858^{(8)}$	$12.8012^{(7)}$	$31.1235^{(12)}$	$0.0229^{(1)}$	$30.8056^{(10)}$	$0.0494^{(3)}$	$0.0481^{(2)}$	$36.0816^{(14)}$	$30.8101^{(11)}$	$32.7835^{(13)}$
	$MSE(\hat{c})$	$3.227^{(6)}$	$5.0996^{(10)}$	$13.7455^{(15)}$	$7.2129^{(14)}$	$6.8343^{(13)}$	$6.7766^{(12)}$	$5.5033^{(11)}$	$2.863^{(4)}$	$0.0191^{(1)}$	$3.1263^{(5)}$	$0.1066^{(2)}$	$0.8491^{(3)}$	$4.5164^{(8)}$	$4.833^{(9)}$	$4.3191^{(7)}$
10	$MSE(\hat{k})$	$281.6467^{(15)}$	$177.9079^{(10)}$	$191.4408^{(13)}$	$189.8471^{(12)}$	$178.3853^{(11)}$	$154.1223^{(5)}$	$168.5305^{(7)}$	$132.6906^{(4)}$	$0.003^{(1)}$	$192.8548^{(14)}$	$0.0268^{(2)}$	$0.1332^{(3)}$	$163.812^{(6)}$	$172.645^{(9)}$	$171.8906^{(8)}$
	$MRE(\hat{\delta})$	$0.6581^{(4)}$	$0.7844^{(13)}$	$0.6697^{(5)}$	$0.7905^{(15)}$	$0.7288^{(8)}$	$0.7532^{(10)}$	$0.7723^{(12)}$	$0.7062^{(6)}$	$0.2231^{(1)}$	$0.7603^{(11)}$	$0.3386^{(2)}$	$0.4093^{(3)}$	$0.7441^{(9)}$	$0.7888^{(14)}$	$0.725^{(7)}$
	$MRE(\hat{\lambda})$	$6.2999^{(15)}$	$2.1477^{(5)}$	$1.8751^{(4)}$	$2.7857^{(9)}$	$2.3088^{(6)}$	$2.4458^{(7)}$	$2.6202^{(8)}$	$3.4346^{(10)}$	$0.1456^{(1)}$	$3.7601^{(12)}$	$0.2117^{(3)}$	$0.2067^{(2)}$	$4.0843^{(14)}$	$3.5452^{(11)}$	$3.9498^{(13)}$
	$MRE(\hat{c})$	$0.6487^{(8)}$	$0.8067^{(13)}$	$0.9941^{(15)}$	$0.8743^{(14)}$	$0.7666^{(9)}$	$0.7971^{(12)}$	$0.7812^{(10)}$	$0.6366^{(6)}$	$0.0262^{(1)}$	$0.6449^{(7)}$	$0.0666^{(2)}$	$0.2196^{(3)}$	$0.6357^{(5)}$	$0.7834^{(11)}$	$0.6311^{(4)}$
	$MRE(\hat{k})$	$6.4832^{(15)}$	$4.3316^{(10)}$	$4.6265^{(13)}$	$4.5192^{(12)}$	$4.3685^{(11)}$	$3.7659^{(5)}$	$4.1696^{(7)}$	$3.2796^{(4)}$	$0.0134^{(1)}$	$4.6506^{(14)}$	$0.031^{(2)}$	$0.071^{(3)}$	$4.0792^{(6)}$	$4.2127^{(8)}$	$4.2532^{(9)}$
	D_{abs}	$0.0911^{(11)}$	$0.083^{(3)}$	$0.0859^{(8)}$	$0.0842^{(5)}$	$0.0855^{(7)}$	$0.0844^{(6)}$	$0.084^{(4)}$	$0.0876^{(10)}$	$0.0789^{(1)}$	$0.0829^{(2)}$	$0.0975^{(14)}$	$0.098^{(15)}$	$0.0942^{(12)}$	$0.086^{(9)}$	$0.0947^{(13)}$
	D_{max}	$0.1789^{(15)}$	$0.1494^{(6)}$	$0.1625^{(13)}$	$0.1523^{(8)}$	$0.1645^{(14)}$	$0.1472^{(5)}$	$0.1466^{(4)}$	$0.1603^{(11)}$	$0.1217^{(1)}$	$0.1422^{(2)}$	$0.1446^{(3)}$	$0.1497^{(7)}$	$0.1595^{(10)}$	$0.1527^{(9)}$	$0.1606^{(12)}$
	$\sum$ Ranks	$146^{(13)}$	$130^{(8)}$	$132^{(11)}$	$161^{(15)}$	$121^{(7)}$	$114^{(5)}$	$120^{(6)}$	$94^{(4)}$	$19^{(1)}$	$132^{(11)}$	$45^{(2)}$	$57^{(3)}$	$132^{(11)}$	$147^{(14)}$	$130^{(8)}$
	$\lvert Bias\rvert(\hat{\delta})$	$0.0121^{(2)}$	$0.1313^{(11)}$	$0.0728^{(7)}$	$0.121^{(10)}$	$0.0667^{(6)}$	$0.0967^{(8)}$	$0.1322^{(12)}$	$0.0664^{(5)}$	$0.0076^{(1)}$	$0.1417^{(15)}$	$0.0178^{(3)}$	$0.0494^{(4)}$	$0.1375^{(13)}$	$0.1087^{(9)}$	$0.1376^{(14)}$
	$\lvert Bias\rvert(\hat{\lambda})$	$2.6331^{(15)}$	$0.6825^{(7)}$	$0.4956^{(4)}$	$1.2214^{(12)}$	$0.5552^{(5)}$	$0.6139^{(6)}$	$0.7436^{(8)}$	$1.0791^{(10)}$	$0.0013^{(1)}$	$1.1006^{(11)}$	$0.06^{(2)}$	$0.0862^{(3)}$	$2.3662^{(13)}$	$0.9635^{(9)}$	$2.4087^{(14)}$
	$\lvert Bias\rvert(\hat{c})$	$0.5462^{(4)}$	$1.8795^{(14)}$	$1.0297^{(7)}$	$2.0475^{(15)}$	$0.9994^{(6)}$	$1.0966^{(8)}$	$1.5669^{(9)}$	$0.9785^{(5)}$	$0.0021^{(1)}$	$1.8184^{(13)}$	$0.0582^{(2)}$	$0.1484^{(3)}$	$1.6342^{(11)}$	$1.7656^{(12)}$	$1.6265^{(10)}$
	$\lvert Bias\rvert(\hat{k})$	$11.2465^{(15)}$	$3.5425^{(5)}$	$3.628^{(6)}$	$6.4324^{(13)}$	$3.6888^{(7)}$	$3.4431^{(4)}$	$3.864^{(8)}$	$6.5384^{(14)}$	$3e-04^{(1)}$	$4.1711^{(9)}$	$0.0472^{(3)}$	$0.0304^{(2)}$	$5.7273^{(11)}$	$5.1391^{(10)}$	$5.761^{(12)}$
	$MSE(\hat{\delta})$	$0.0224^{(4)}$	$0.0435^{(11)}$	$0.0419^{(9)}$	$0.0434^{(10)}$	$0.0366^{(5)}$	$0.0407^{(8)}$	$0.0453^{(12)}$	$0.0369^{(6)}$	$0.0059^{(1)}$	$0.0473^{(13)}$	$0.0102^{(2)}$	$0.0202^{(3)}$	$0.0487^{(14)}$	$0.0395^{(7)}$	$0.0487^{(15)}$
	$MSE(\hat{\lambda})$	$17.3331^{(13)}$	$2.6324^{(6)}$	$1.6457^{(4)}$	$6.1907^{(11)}$	$2.3582^{(5)}$	$2.9182^{(7)}$	$3.2631^{(8)}$	$4.2118^{(10)}$	$0.0276^{(1)}$	$6.9531^{(12)}$	$0.0683^{(3)}$	$0.047^{(2)}$	$29.506^{(14)}$	$3.6367^{(9)}$	$30.368^{(15)}$
	$MSE(\hat{c})$	$7.5111^{(9)}$	$10.1365^{(10)}$	$3.4508^{(5)}$	$14.8837^{(13)}$	$3.3629^{(4)}$	$3.6663^{(6)}$	$6.1734^{(7)}$	$6.5758^{(8)}$	$0.0024^{(1)}$	$11.3656^{(12)}$	$0.1378^{(2)}$	$0.7161^{(3)}$	$17.4003^{(15)}$	$10.881^{(11)}$	$17.3732^{(14)}$
20	$MSE(\hat{k})$	$202.2388^{(15)}$	$72.7346^{(7)}$	$70.891^{(6)}$	$123.9143^{(14)}$	$70.8238^{(5)}$	$70.7937^{(4)}$	$74.4694^{(8)}$	$117.4348^{(13)}$	$0.0016^{(1)}$	$82.8522^{(9)}$	$0.3343^{(3)}$	$0.2187^{(2)}$	$102.1558^{(11)}$	$101.6824^{(10)}$	$102.4981^{(12)}$
	$MRE(\hat{\delta})$	$0.4061^{(4)}$	$0.5902^{(10)}$	$0.5863^{(9)}$	$0.5922^{(11)}$	$0.5392^{(6)}$	$0.578^{(8)}$	$0.6136^{(12)}$	$0.5281^{(5)}$	$0.1959^{(1)}$	$0.6363^{(13)}$	$0.2426^{(2)}$	$0.3409^{(3)}$	$0.6609^{(15)}$	$0.5573^{(7)}$	$0.6605^{(14)}$
	$MRE(\hat{\lambda})$	$3.3994^{(15)}$	$1.0942^{(5)}$	$1.0251^{(4)}$	$1.7207^{(12)}$	$1.142^{(6)}$	$1.2232^{(7)}$	$1.2705^{(8)}$	$1.4681^{(10)}$	$0.1642^{(1)}$	$1.6066^{(11)}$	$0.2091^{(2)}$	$0.2095^{(3)}$	$3.178^{(13)}$	$1.4156^{(9)}$	$3.2319^{(14)}$
	$MRE(\hat{c})$	$0.7103^{(7)}$	$1.0225^{(12)}$	$0.6245^{(5)}$	$1.1368^{(15)}$	$0.6015^{(4)}$	$0.6344^{(6)}$	$0.8342^{(9)}$	$0.8066^{(8)}$	$0.0137^{(1)}$	$1.0413^{(13)}$	$0.0882^{(2)}$	$0.2568^{(3)}$	$1.0092^{(11)}$	$1.0444^{(14)}$	$1.0063^{(10)}$
	$MRE(\hat{k})$	$4.792^{(15)}$	$2.1053^{(7)}$	$2.0784^{(6)}$	$3.1343^{(14)}$	$2.0574^{(4)}$	$2.0728^{(5)}$	$2.2975^{(8)}$	$2.9284^{(13)}$	$0.0118^{(1)}$	$2.4443^{(9)}$	$0.0629^{(2)}$	$0.1158^{(3)}$	$2.8859^{(11)}$	$2.7119^{(10)}$	$2.8978^{(12)}$
	D_{abs}	$0.069^{(7)}$	$0.0683^{(5)}$	$0.0715^{(12)}$	$0.0702^{(8)}$	$0.0708^{(10)}$	$0.0705^{(9)}$	$0.0684^{(6)}$	$0.0738^{(13)}$	$0.0658^{(2)}$	$0.0675^{(4)}$	$0.0666^{(3)}$	$0.0623^{(1)}$	$0.0783^{(14)}$	$0.0709^{(11)}$	$0.0785^{(15)}$
	D_{max}	$0.1229^{(7)}$	$0.1204^{(5)}$	$0.131^{(12)}$	$0.1266^{(10)}$	$0.1297^{(11)}$	$0.1254^{(9)}$	$0.1211^{(6)}$	$0.1351^{(13)}$	$0.1047^{(2)}$	$0.1146^{(4)}$	$0.1068^{(3)}$	$0.101^{(1)}$	$0.1375^{(14)}$	$0.1245^{(8)}$	$0.1378^{(15)}$
	$\sum$ Ranks	$132^{(9)}$	$115^{(7)}$	$96^{(6)}$	$168^{(13)}$	$84^{(4)}$	$95^{(5)}$	$121^{(8)}$	$133^{(10)}$	$16^{(1)}$	$148^{(12)}$	$34^{(2)}$	$36^{(3)}$	$180^{(14)}$	$136^{(11)}$	$186^{(15)}$
	$\lvert Bias\rvert(\hat{\delta})$	$0.0228^{(3)}$	$0.0779^{(8)}$	$0.0782^{(9)}$	$0.0661^{(6)}$	$0.096^{(11)}$	$0.1183^{(13)}$	$0.0965^{(12)}$	$0.0349^{(5)}$	$0.0061^{(1)}$	$0.0916^{(10)}$	$0.0118^{(2)}$	$0.0323^{(4)}$	$0.1205^{(14)}$	$0.0662^{(7)}$	$0.1284^{(15)}$
	$\lvert Bias\rvert(\hat{\lambda})$	$1.4165^{(15)}$	$0.6431^{(7)}$	$0.6639^{(9)}$	$0.8575^{(13)}$	$0.8337^{(12)}$	$0.8324^{(11)}$	$0.6887^{(10)}$	$1.0769^{(14)}$	$0.0016^{(1)}$	$0.5927^{(6)}$	$0.0157^{(2)}$	$0.0256^{(3)}$	$0.5371^{(4)}$	$0.652^{(8)}$	$0.5453^{(5)}$
	$\lvert Bias\rvert(\hat{c})$	$0.0996^{(3)}$	$0.7569^{(8)}$	$0.7499^{(7)}$	$0.7175^{(6)}$	$0.8698^{(12)}$	$0.9726^{(15)}$	$0.8469^{(11)}$	$0.5859^{(5)}$	$0.0113^{(2)}$	$0.8735^{(13)}$	$0.0067^{(1)}$	$0.1761^{(4)}$	$0.8448^{(10)}$	$0.7723^{(9)}$	$0.9308^{(14)}$
	$\lvert Bias\rvert(\hat{k})$	$8.9794^{(15)}$	$4.4993^{(8)}$	$4.6178^{(10)}$	$5.8123^{(13)}$	$5.3197^{(12)}$	$4.9275^{(11)}$	$4.6152^{(9)}$	$6.6084^{(14)}$	$0.0013^{(1)}$	$3.7665^{(6)}$	$0.0225^{(2)}$	$0.1114^{(3)}$	$3.0508^{(5)}$	$4.3191^{(7)}$	$3.0392^{(4)}$
	$MSE(\hat{\delta})$	$0.0065^{(3)}$	$0.0258^{(9)}$	$0.0288^{(10)}$	$0.0231^{(7)}$	$0.0349^{(14)}$	$0.0379^{(15)}$	$0.0307^{(11)}$	$0.0161^{(5)}$	$0.0014^{(1)}$	$0.0254^{(8)}$	$0.0043^{(2)}$	$0.0076^{(4)}$	$0.0332^{(12)}$	$0.0214^{(6)}$	$0.0347^{(13)}$
	$MSE(\hat{\lambda})$	$4.5328^{(15)}$	$1.5456^{(6)}$	$1.8872^{(10)}$	$2.387^{(13)}$	$2.2245^{(11)}$	$2.3852^{(12)}$	$1.7494^{(8)}$	$4.4486^{(14)}$	$0.0064^{(1)}$	$1.6425^{(7)}$	$0.0143^{(2)}$	$0.0165^{(3)}$	$1.3355^{(4)}$	$1.8046^{(9)}$	$1.3429^{(5)}$
	$MSE(\hat{c})$	$0.8384^{(4)}$	$2.6883^{(6)}$	$2.0689^{(5)}$	$3.6978^{(14)}$	$3.3236^{(12)}$	$3.3835^{(13)}$	$3.0095^{(10)}$	$4.4196^{(15)}$	$0.0016^{(1)}$	$2.9549^{(9)}$	$0.1539^{(2)}$	$0.6171^{(3)}$	$2.7541^{(7)}$	$2.8443^{(8)}$	$3.1159^{(11)}$
50	$MSE(\hat{k})$	$152.0839^{(15)}$	$84.2678^{(7)}$	$85.2974^{(9)}$	$103.8398^{(13)}$	$98.2981^{(12)}$	$95.4834^{(11)}$	$88.6081^{(10)}$	$110.7018^{(14)}$	$5e-04^{(1)}$	$76.1241^{(6)}$	$0.0586^{(2)}$	$0.2437^{(3)}$	$62.0939^{(5)}$	$85.132^{(8)}$	$62.0153^{(4)}$
	$MRE(\hat{\delta})$	$0.2157^{(4)}$	$0.3925^{(8)}$	$0.459^{(11)}$	$0.374^{(7)}$	$0.4919^{(13)}$	$0.5209^{(15)}$	$0.451^{(10)}$	$0.3005^{(5)}$	$0.0941^{(1)}$	$0.4061^{(9)}$	$0.1536^{(2)}$	$0.2033^{(3)}$	$0.488^{(12)}$	$0.367^{(6)}$	$0.5015^{(14)}$
	$MRE(\hat{\lambda})$	$1.9242^{(15)}$	$0.9493^{(7)}$	$1.0881^{(10)}$	$1.2614^{(13)}$	$1.1854^{(11)}$	$1.1965^{(12)}$	$1.0189^{(8)}$	$1.4922^{(14)}$	$0.0766^{(1)}$	$0.9478^{(6)}$	$0.1129^{(2)}$	$0.1237^{(3)}$	$0.8763^{(5)}$	$1.0319^{(9)}$	$0.8742^{(4)}$
	$MRE(\hat{c})$	$0.3023^{(4)}$	$0.523^{(6)}$	$0.4842^{(5)}$	$0.5235^{(7)}$	$0.5807^{(13)}$	$0.6077^{(15)}$	$0.5526^{(10)}$	$0.5383^{(8)}$	$0.0101^{(1)}$	$0.5622^{(12)}$	$0.0934^{(2)}$	$0.1957^{(3)}$	$0.5568^{(11)}$	$0.5434^{(9)}$	$0.5931^{(14)}$
	$MRE(\hat{k})$	$3.8411^{(15)}$	$2.3019^{(7)}$	$2.405^{(10)}$	$2.712^{(13)}$	$2.5988^{(12)}$	$2.5232^{(11)}$	$2.3887^{(9)}$	$2.7197^{(14)}$	$0.0062^{(1)}$	$2.1501^{(6)}$	$0.0369^{(2)}$	$0.0981^{(3)}$	$1.9244^{(4)}$	$2.3129^{(8)}$	$1.9291^{(5)}$
	D_{abs}	$0.0397^{(5)}$	$0.0387^{(2)}$	$0.0406^{(9)}$	$0.0407^{(10)}$	$0.0402^{(7)}$	$0.0403^{(8)}$	$0.0388^{(4)}$	$0.0427^{(13)}$	$0.0379^{(1)}$	$0.0388^{(3)}$	$0.0423^{(12)}$	$0.0421^{(11)}$	$0.0486^{(14)}$	$0.0402^{(6)}$	$0.0488^{(15)}$
	D_{max}	$0.0683^{(7)}$	$0.0672^{(3)}$	$0.0743^{(13)}$	$0.0708^{(9)}$	$0.0713^{(11)}$	$0.0707^{(8)}$	$0.0677^{(4)}$	$0.0742^{(12)}$	$0.0595^{(1)}$	$0.0663^{(2)}$	$0.0682^{(5)}$	$0.0682^{(6)}$	$0.0823^{(14)}$	$0.071^{(10)}$	$0.0827^{(15)}$
	$\sum$ Ranks	$123^{(8)}$	$92^{(4)}$	$127^{(10)}$	$144^{(12)}$	$163^{(14)}$	$170^{(15)}$	$126^{(9)}$	$152^{(13)}$	$15^{(1)}$	$103^{(5)}$	$40^{(2)}$	$56^{(3)}$	$121^{(7)}$	$110^{(6)}$	$138^{(11)}$
	$\lvert Bias\rvert(\hat{\delta})$	$0.0118^{(3)}$	$0.0555^{(9)}$	$0.048^{(8)}$	$0.0447^{(7)}$	$0.0562^{(10)}$	$0.0621^{(11)}$	$0.0623^{(12)}$	$0.018^{(4)}$	$9e-04^{(1)}$	$0.0636^{(13)}$	$0.0108^{(2)}$	$0.0268^{(5)}$	$0.092^{(15)}$	$0.0375^{(6)}$	$0.0918^{(14)}$
	$\lvert Bias\rvert(\hat{\lambda})$	$1.0591^{(15)}$	$0.3826^{(5)}$	$0.4947^{(10)}$	$0.6448^{(13)}$	$0.5339^{(12)}$	$0.4658^{(9)}$	$0.4259^{(8)}$	$0.811^{(14)}$	$0.012^{(1)}$	$0.3147^{(4)}$	$0.0193^{(2)}$	$0.0344^{(3)}$	$0.4117^{(6)}$	$0.5321^{(11)}$	$0.4147^{(7)}$
	$\lvert Bias\rvert(\hat{c})$	$0.0358^{(3)}$	$0.571^{(11)}$	$0.4257^{(6)}$	$0.4574^{(8)}$	$0.5217^{(10)}$	$0.5215^{(9)}$	$0.5986^{(12)}$	$0.2705^{(5)}$	$0.0085^{(1)}$	$0.6591^{(13)}$	$0.0135^{(2)}$	$0.2256^{(4)}$	$0.7586^{(15)}$	$0.4402^{(7)}$	$0.7553^{(14)}$
	$\lvert Bias\rvert(\hat{k})$	$6.6341^{(15)}$	$2.0377^{(7)}$	$2.8009^{(9)}$	$4.4151^{(13)}$	$3.5433^{(12)}$	$2.9413^{(10)}$	$2.496^{(8)}$	$5.5616^{(14)}$	$0.0078^{(1)}$	$1.5493^{(4)}$	$0.0117^{(2)}$	$0.0519^{(3)}$	$1.9693^{(5)}$	$3.1897^{(11)}$	$1.9843^{(6)}$
	$MSE(\hat{\delta})$	$0.0051^{(4)}$	$0.0158^{(8)}$	$0.0201^{(13)}$	$0.0133^{(7)}$	$0.0193^{(11)}$	$0.0197^{(12)}$	$0.0172^{(10)}$	$0.008^{(5)}$	$0.0013^{(1)}$	$0.0163^{(9)}$	$0.002^{(2)}$	$0.005^{(3)}$	$0.0236^{(15)}$	$0.0114^{(6)}$	$0.0235^{(14)}$
	$MSE(\hat{\lambda})$	$2.7465^{(15)}$	$0.9102^{(5)}$	$1.3522^{(12)}$	$1.6178^{(13)}$	$1.2573^{(11)}$	$1.1488^{(9)}$	$1.0339^{(8)}$	$2.3521^{(14)}$	$0.0053^{(1)}$	$0.7451^{(4)}$	$0.0062^{(2)}$	$0.0533^{(3)}$	$1.0024^{(6)}$	$1.2554^{(10)}$	$1.0122^{(7)}$
	$MSE(\hat{c})$	$0.8628^{(4)}$	$2.0597^{(10)}$	$1.172^{(5)}$	$1.5337^{(6)}$	$2.0776^{(11)}$	$1.7819^{(9)}$	$2.222^{(12)}$	$1.58^{(7)}$	$0.0011^{(1)}$	$2.2901^{(13)}$	$0.0758^{(2)}$	$0.5139^{(3)}$	$2.8535^{(15)}$	$1.6409^{(8)}$	$2.8526^{(14)}$
100	$MSE(\hat{k})$	$107.6665^{(15)}$	$34.9041^{(5)}$	$50.0041^{(9)}$	$78.4949^{(13)}$	$63.1301^{(12)}$	$53.6324^{(11)}$	$46.2587^{(8)}$	$98.066^{(14)}$	$9e-04^{(1)}$	$29.1265^{(4)}$	$0.0163^{(2)}$	$1.6621^{(3)}$	$38.6119^{(6)}$	$52.7762^{(10)}$	$38.7876^{(7)}$
	$MRE(\hat{\delta})$	$0.1701^{(4)}$	$0.2911^{(8)}$	$0.3325^{(12)}$	$0.2616^{(7)}$	$0.3297^{(11)}$	$0.3335^{(13)}$	$0.3047^{(9)}$	$0.2056^{(5)}$	$0.0857^{(1)}$	$0.3085^{(10)}$	$0.1159^{(2)}$	$0.1663^{(3)}$	$0.3749^{(15)}$	$0.2504^{(6)}$	$0.3737^{(14)}$
	$MRE(\hat{\lambda})$	$1.4403^{(15)}$	$0.6165^{(5)}$	$0.8278^{(11)}$	$1.0004^{(13)}$	$0.7847^{(10)}$	$0.7211^{(9)}$	$0.655^{(6)}$	$1.1787^{(14)}$	$0.0711^{(1)}$	$0.5802^{(4)}$	$0.0737^{(2)}$	$0.1414^{(3)}$	$0.7026^{(7)}$	$0.8525^{(12)}$	$0.7067^{(8)}$
	$MRE(\hat{c})$	$0.2904^{(4)}$	$0.4093^{(10)}$	$0.3575^{(6)}$	$0.3753^{(7)}$	$0.4172^{(11)}$	$0.403^{(9)}$	$0.4228^{(12)}$	$0.3553^{(5)}$	$0.0076^{(1)}$	$0.4601^{(13)}$	$0.0673^{(2)}$	$0.177^{(3)}$	$0.5261^{(15)}$	$0.396^{(8)}$	$0.5246^{(14)}$
	$MRE(\hat{k})$	$2.9^{(15)}$	$1.2374^{(5)}$	$1.5857^{(9)}$	$2.2336^{(13)}$	$1.7922^{(12)}$	$1.5968^{(10)}$	$1.4351^{(8)}$	$2.5827^{(14)}$	$0.0068^{(1)}$	$1.177^{(4)}$	$0.0239^{(2)}$	$0.1806^{(3)}$	$1.3899^{(6)}$	$1.758^{(11)}$	$1.3904^{(7)}$
	D_{abs}	$0.0271^{(5)}$	$0.0268^{(2)}$	$0.0283^{(9)}$	$0.0278^{(7)}$	$0.0278^{(6)}$	$0.028^{(8)}$	$0.0269^{(3)}$	$0.0296^{(11)}$	$0.0258^{(1)}$	$0.027^{(4)}$	$0.03^{(12)}$	$0.0294^{(10)}$	$0.0329^{(14)}$	$0.0319^{(13)}$	$0.033^{(15)}$
	D_{max}	$0.0473^{(3)}$	$0.048^{(4)}$	$0.0524^{(12)}$	$0.0504^{(8)}$	$0.0509^{(9)}$	$0.0509^{(10)}$	$0.0481^{(5)}$	$0.0521^{(11)}$	$0.0418^{(1)}$	$0.0469^{(2)}$	$0.0481^{(6)}$	$0.0492^{(7)}$	$0.0571^{(14)}$	$0.0558^{(13)}$	$0.0573^{(15)}$
	$\sum$ Ranks	$120^{(6)}$	$94^{(4)}$	$131^{(8)}$	$135^{(10)}$	$148^{(13)}$	$139^{(12)}$	$121^{(7)}$	$137^{(11)}$	$14^{(1)}$	$101^{(5)}$	$42^{(2)}$	$56^{(3)}$	$154^{(14)}$	$132^{(9)}$	$156^{(15)}$
	$\lvert Bias\rvert(\hat{\delta})$	$0.0058^{(2)}$	$0.0314^{(11)}$	$0.0109^{(3)}$	$0.0367^{(12)}$	$0.017^{(5)}$	$0.0249^{(8)}$	$0.0267^{(9)}$	$0.0198^{(6)}$	$0.0015^{(1)}$	$0.0401^{(13)}$	$0.0111^{(4)}$	$0.0233^{(7)}$	$0.0679^{(14)}$	$0.0309^{(10)}$	$0.0682^{(15)}$
	$\lvert Bias\rvert(\hat{\lambda})$	$0.453^{(13)}$	$0.3005^{(9)}$	$0.4276^{(12)}$	$0.3045^{(10)}$	$0.507^{(15)}$	$0.4817^{(14)}$	$0.3155^{(11)}$	$0.2296^{(8)}$	$0.0021^{(1)}$	$0.1048^{(4)}$	$0.0093^{(2)}$	$0.0126^{(3)}$	$0.1885^{(5)}$	$0.1967^{(7)}$	$0.193^{(6)}$
	$\lvert Bias\rvert(\hat{c})$	$0.0514^{(3)}$	$0.3173^{(10)}$	$0.0939^{(4)}$	$0.4138^{(13)}$	$0.158^{(5)}$	$0.2057^{(7)}$	$0.2378^{(8)}$	$0.2708^{(9)}$	$1e-04^{(1)}$	$0.3959^{(12)}$	$0.0281^{(2)}$	$0.2038^{(6)}$	$0.5401^{(15)}$	$0.3726^{(11)}$	$0.5391^{(14)}$
	$\lvert Bias\rvert(\hat{k})$	$3.0749^{(13)}$	$1.8103^{(8)}$	$2.8479^{(12)}$	$2.1389^{(10)}$	$3.5338^{(15)}$	$3.2611^{(14)}$	$1.8856^{(9)}$	$2.2203^{(11)}$	$0.0081^{(2)}$	$0.49^{(4)}$	$0.0056^{(1)}$	$0.1522^{(3)}$	$0.9392^{(5)}$	$1.0992^{(7)}$	$0.9619^{(6)}$
	$MSE(\hat{\delta})$	$0.0014^{(3)}$	$0.0079^{(9)}$	$0.0088^{(11)}$	$0.0087^{(10)}$	$0.0093^{(12)}$	$0.0111^{(13)}$	$0.0068^{(8)}$	$0.0038^{(5)}$	$6e-04^{(1)}$	$0.0062^{(6)}$	$0.0012^{(2)}$	$0.0023^{(4)}$	$0.0113^{(15)}$	$0.0068^{(7)}$	$0.0113^{(14)}$
	$MSE(\hat{\lambda})$	$0.9617^{(13)}$	$0.6546^{(9)}$	$0.9188^{(12)}$	$0.7267^{(11)}$	$1.1776^{(15)}$	$1.14^{(14)}$	$0.6867^{(10)}$	$0.5317^{(6)}$	$0.0024^{(1)}$	$0.311^{(4)}$	$0.0036^{(2)}$	$0.0109^{(3)}$	$0.5492^{(7)}$	$0.5171^{(5)}$	$0.5578^{(8)}$
	$MSE(\hat{c})$	$0.2058^{(3)}$	$1.0412^{(11)}$	$0.5414^{(5)}$	$1.2386^{(13)}$	$0.7656^{(8)}$	$0.892^{(10)}$	$0.7447^{(7)}$	$0.7247^{(6)}$	$0.0022^{(1)}$	$0.8429^{(9)}$	$0.0553^{(2)}$	$0.328^{(4)}$	$1.2525^{(15)}$	$1.0914^{(12)}$	$1.2516^{(14)}$
200	$MSE(\hat{k})$	$42.9436^{(13)}$	$26.103^{(8)}$	$41.7233^{(12)}$	$36.5816^{(10)}$	$57.3591^{(15)}$	$53.7513^{(14)}$	$26.2513^{(9)}$	$40.5788^{(11)}$	$0.0046^{(1)}$	$10.129^{(4)}$	$0.0051^{(2)}$	$0.2947^{(3)}$	$23.3657^{(6)}$	$18.0572^{(5)}$	$23.6656^{(7)}$
	$MRE(\hat{\delta})$	$0.0886^{(3)}$	$0.1847^{(9)}$	$0.2052^{(11)}$	$0.1911^{(10)}$	$0.2162^{(12)}$	$0.2337^{(13)}$	$0.1837^{(8)}$	$0.1307^{(5)}$	$0.0635^{(1)}$	$0.1834^{(7)}$	$0.0877^{(2)}$	$0.117^{(4)}$	$0.2779^{(15)}$	$0.1648^{(6)}$	$0.2757^{(14)}$
	$MRE(\hat{\lambda})$	$0.7113^{(13)}$	$0.5438^{(9)}$	$0.7012^{(12)}$	$0.6018^{(11)}$	$0.784^{(15)}$	$0.7546^{(14)}$	$0.5602^{(10)}$	$0.5086^{(8)}$	$0.0494^{(1)}$	$0.3566^{(4)}$	$0.0561^{(2)}$	$0.094^{(3)}$	$0.4341^{(6)}$	$0.4689^{(7)}$	$0.4332^{(5)}$

Table 4: Partial and overall ranks for all estimation methods of the SHBXII model from Table 3.

n	Est.	MLE	ADE	RTADE	LTADE	CVME	OLSE	WLSE	ADSOE	KE	MPSE	MSADE	MSALDE	MSSDE	MSSLDE	MSLNDE
	$\lvert Bias\rvert(\hat{\delta})$	6	15	7	13	3	10	14	1	2	9	4	5	11	12	8
	$\lvert Bias\rvert(\hat{\lambda})$	15	5	4	9	6	7	8	10	2	12	3	1	14	11	13
	$\lvert Bias\rvert(\hat{c})$	1	12	15	14	10	13	11	5	3	8	2	4	6	9	7
	$\lvert Bias\rvert(\hat{k})$	15	10	12	13	11	5	6	4	2	14	1	3	7	9	8
	$MSE(\hat{\delta})$	5	13	4	14	6	9	11	7	1	12	2	3	10	15	8
	$MSE(\hat{\lambda})$	15	5	4	9	6	8	7	12	1	10	3	2	14	11	13
	$MSE(\hat{c})$	6	10	15	14	13	12	11	4	1	5	2	3	8	9	7
10	$MSE(\hat{k})$	15	10	13	12	11	5	7	4	1	14	2	3	6	9	8
	$MRE(\hat{\delta})$	4	13	5	15	8	10	12	6	1	11	2	3	9	14	7
	$MRE(\hat{\lambda})$	15	5	4	9	6	7	8	10	1	12	3	2	14	11	13
	$MRE(\hat{c})$	8	13	15	14	9	12	10	6	1	7	2	3	5	11	4
	$MRE(\hat{k})$	15	10	13	12	11	5	7	4	1	14	2	3	6	8	9
	D_{abs}	11	3	8	5	7	6	4	10	1	2	14	15	12	9	13
	D_{max}	15	6	13	8	14	5	4	11	1	2	3	7	10	9	12
	$\sum$ Ranks	$146^{\{13\}}$	$130^{\{8\}}$	$132^{\{11\}}$	$161^{\{15\}}$	$121^{\{7\}}$	$114^{\{5\}}$	$120^{\{6\}}$	$94^{\{4\}}$	$19^{\{1\}}$	$132^{\{11\}}$	$45^{\{2\}}$	$57^{\{3\}}$	$132^{\{11\}}$	$147^{\{14\}}$	$130^{\{8\}}$
	$\lvert Bias\rvert(\hat{\delta})$	2	11	7	10	6	8	12	5	1	15	3	4	13	9	14
	$\lvert Bias\rvert(\hat{\lambda})$	15	7	4	12	5	6	8	10	1	11	2	3	13	9	14
	$\lvert Bias\rvert(\hat{c})$	4	14	7	15	6	8	9	5	1	13	2	3	11	12	10
	$\lvert Bias\rvert(\hat{k})$	15	5	6	13	7	4	8	14	1	9	3	2	11	10	12
	$MSE(\hat{\delta})$	4	11	9	10	5	8	12	6	1	13	2	3	14	7	15
	$MSE(\hat{\lambda})$	13	6	4	11	5	7	8	10	1	12	3	2	14	9	15
	$MSE(\hat{c})$	9	10	5	13	4	6	7	8	1	12	2	3	15	11	14
20	$MSE(\hat{k})$	15	7	6	14	5	4	8	13	1	9	3	2	11	10	12
	$MRE(\hat{\delta})$	4	10	9	11	6	8	12	5	1	13	2	3	15	7	14
	$MRE(\hat{\lambda})$	15	5	4	12	6	7	8	10	1	11	2	3	13	9	14
	$MRE(\hat{c})$	7	12	5	15	4	6	9	8	1	13	2	3	11	14	10
	$MRE(\hat{k})$	15	7	6	14	4	5	8	13	1	9	2	3	11	10	12
	D_{abs}	7	5	12	8	10	9	6	13	2	4	3	1	14	11	15
	D_{max}	7	5	12	10	11	9	6	13	2	4	3	1	14	8	15
	$\sum$ Ranks	$132^{\{9\}}$	$115^{\{7\}}$	$96^{\{6\}}$	$168^{\{13\}}$	$84^{\{4\}}$	$95^{\{5\}}$	$121^{\{8\}}$	$133^{\{10\}}$	$16^{\{1\}}$	$148^{\{12\}}$	$34^{\{2\}}$	$36^{\{3\}}$	$180^{\{14\}}$	$136^{\{11\}}$	$186^{\{15\}}$
	$\lvert Bias\rvert(\hat{\delta})$	3	8	9	6	11	13	12	5	1	10	2	4	14	7	15
	$\lvert Bias\rvert(\hat{\lambda})$	15	7	9	13	12	11	10	14	1	6	2	3	4	8	5
	$\lvert Bias\rvert(\hat{c})$	3	8	7	6	12	15	11	5	2	13	1	4	10	9	14
	$\lvert Bias\rvert(\hat{k})$	15	8	10	13	12	11	9	14	1	6	2	3	5	7	4
	$MSE(\hat{\delta})$	3	9	10	7	14	15	11	5	1	8	2	4	12	6	13
	$MSE(\hat{\lambda})$	15	6	10	13	11	12	8	14	1	7	2	3	4	9	5
	$MSE(\hat{c})$	4	6	5	14	12	13	10	15	1	9	2	3	7	8	11
50	$MSE(\hat{k})$	15	7	9	13	12	11	10	14	1	6	2	3	5	8	4
	$MRE(\hat{\delta})$	4	8	11	7	13	15	10	5	1	9	2	3	12	6	14
	$MRE(\hat{\lambda})$	15	7	10	13	11	12	8	14	1	6	2	3	5	9	4
	$MRE(\hat{c})$	4	6	5	7	13	15	10	8	1	12	2	3	11	9	14
	$MRE(\hat{k})$	15	7	10	13	12	11	9	14	1	6	2	3	4	8	5
	D_{abs}	5	2	9	10	7	8	4	13	1	3	12	11	14	6	15
	D_{max}	7	3	13	9	11	8	4	12	1	2	5	6	14	10	15
	$\sum$ Ranks	$123^{\{8\}}$	$92^{\{4\}}$	$127^{\{10\}}$	$144^{\{12\}}$	$163^{\{14\}}$	$170^{\{15\}}$	$126^{\{9\}}$	$152^{\{13\}}$	$15^{\{1\}}$	$103^{\{5\}}$	$40^{\{2\}}$	$56^{\{3\}}$	$121^{\{7\}}$	$110^{\{6\}}$	$138^{\{11\}}$
	$\lvert Bias\rvert(\hat{\delta})$	3	9	8	7	10	11	12	4	1	13	2	5	15	6	14
	$\lvert Bias\rvert(\hat{\lambda})$	15	5	10	13	12	9	8	14	1	4	2	3	6	11	7
	$\lvert Bias\rvert(\hat{c})$	3	11	6	8	10	9	12	5	1	13	2	4	15	7	14
	$\lvert Bias\rvert(\hat{k})$	15	7	9	13	12	10	8	14	1	4	2	3	5	11	6
	$MSE(\hat{\delta})$	4	8	13	7	11	12	10	5	1	9	2	3	15	6	14
	$MSE(\hat{\lambda})$	15	5	12	13	11	9	8	14	1	4	2	3	6	10	7
	$MSE(\hat{c})$	4	10	5	6	11	9	12	7	1	13	2	3	15	8	14
100	$MSE(\hat{k})$	15	5	9	13	12	11	8	14	1	4	2	3	6	10	7
	$MRE(\hat{\delta})$	4	8	12	7	11	13	9	5	1	10	2	3	15	6	14
	$MRE(\hat{\lambda})$	15	5	11	13	10	9	6	14	1	4	2	3	7	12	8
	$MRE(\hat{c})$	4	10	6	7	11	9	12	5	1	13	2	3	15	8	14
	$MRE(\hat{k})$	15	5	9	13	12	10	8	14	1	4	2	3	6	11	7
	D_{abs}	5	2	9	7	6	8	3	11	1	4	12	10	14	13	15
	D_{max}	3	4	12	8	9	10	5	11	1	2	6	7	14	13	15
	$\sum$ Ranks	$120^{\{6\}}$	$94^{\{4\}}$	$131^{\{8\}}$	$135^{\{10\}}$	$148^{\{13\}}$	$139^{\{12\}}$	$121^{\{7\}}$	$137^{\{11\}}$	$14^{\{1\}}$	$101^{\{5\}}$	$42^{\{2\}}$	$56^{\{3\}}$	$154^{\{14\}}$	$132^{\{9\}}$	$156^{\{15\}}$
	$\lvert Bias\rvert(\hat{\delta})$	2	11	3	12	5	8	9	6	1	13	4	7	14	10	15
	$\lvert Bias\rvert(\hat{\lambda})$	13	9	12	10	15	14	11	8	1	4	2	3	5	7	6
	$\lvert Bias\rvert(\hat{c})$	3	10	4	13	5	7	8	9	1	12	2	6	15	11	14
	$\lvert Bias\rvert(\hat{k})$	13	8	12	10	15	14	9	11	2	4	1	3	5	7	6
	$MSE(\hat{\delta})$	3	9	11	10	12	13	8	5	1	6	2	4	15	7	14
	$MSE(\hat{\lambda})$	13	9	12	11	15	14	10	6	1	4	2	3	7	5	8
	$MSE(\hat{c})$	3	11	5	13	8	10	7	6	1	9	2	4	15	12	14
200	$MSE(\hat{k})$	13	8	12	10	15	14	9	11	1	4	2	3	6	5	7
	$MRE(\hat{\delta})$	3	9	11	10	12	13	8	5	1	7	2	4	15	6	14
	$MRE(\hat{\lambda})$	13	9	12	11	15	14	10	8	1	4	2	3	6	7	5
	$MRE(\hat{c})$	4	12	5	13	8	11	7	6	1	9	2	3	15	10	14
	$MRE(\hat{k})$	13	8	12	11	15	14	9	10	1	4	2	3	7	5	6
	D_{abs}	2	5	8	9	6	7	4	12	1	3	13	11	14	10	15
	D_{max}	2	5	9	8	7	6	4	13	1	3	12	10	14	11	15
	$\sum$ Ranks	$100^{\{5\}}$	$123^{\{9\}}$	$128^{\{10\}}$	$151^{\{11\}}$	$153^{\{13\}}$	$159^{\{15\}}$	$113^{\{6\}}$	$116^{\{8\}}$	$15^{\{1\}}$	$86^{\{4\}}$	$50^{\{2\}}$	$67^{\{3\}}$	$153^{\{13\}}$	$113^{\{6\}}$	$153^{\{13\}}$
	$\lvert Bias\rvert(\hat{\delta})$	2	10	13	6	11	12	8	3	1	9	4	5	15	7	14
	$\lvert Bias\rvert(\hat{\lambda})$	13	8	9	12	14	11	10	15	2	4	3	1	6	5	7
	$\lvert Bias\rvert(\hat{c})$	4	10	13	5	9	12	7	2	1	11	3	6	15	8	14
	$\lvert Bias\rvert(\hat{k})$	12	9	8	14	13	11	10	15	1	4	2	3	6	5	7
	$MSE(\hat{\delta})$	4	10	15	8	11	14	9	3	1	7	2	5	13	6	12
	$MSE(\hat{\lambda})$	11	6	8	13	14	12	7	15	1	4	3	2	10	5	9
	$MSE(\hat{c})$	4	10	15	8	11	12	9	5	1	6	2	3	14	7	13
300	$MSE(\hat{k})$	11	7	6	14	13	12	10	15	1	4	3	2	9	5	8
	$MRE(\hat{\delta})$	2	10	15	7	11	14	9	4	1	8	3	5	13	6	12
	$MRE(\hat{\lambda})$	12	6	9	13	14	11	10	15	1	4	3	2	8	5	7
	$MRE(\hat{c})$	5	10	15	8	11	12	9	4	1	7	2	3	14	6	13
	$MRE(\hat{k})$	11	8	6	13	14	12	10	15	1	4	2	3	9	5	7
	D_{abs}	3	4	7	6	8	9	5	11	1	2	13	10	15	12	14
	D_{max}	3	4	7	12	9	10	5	13	1	2	8	6	15	11	14
	$\sum$ Ranks	$97^{\{6\}}$	$112^{\{7\}}$	$146^{\{11\}}$	$139^{\{10\}}$	$163^{\{14\}}$	$164^{\{15\}}$	$118^{\{8\}}$	$135^{\{9\}}$	$15^{\{1\}}$	$76^{\{4\}}$	$53^{\{2\}}$	$56^{\{3\}}$	$162^{\{13\}}$	$93^{\{5\}}$	$151^{\{12\}}$

Table 5: Results from simulation Case III.

n	Est.	MLE	ADE	RTADE	LTADE	CVME	OLSE	WLSE	ADSOE	KE	MPSE	MSADE	MSALDE	MSSDE	MSSLDE	MSLNDE
	$\lvert Bias\rvert(\hat{\delta})$	0.0559[8]	0.0581[9]	0.0994[14]	0.0393[6]	0.0865[12]	0.101[15]	0.0915[13]	$8e-04$[1]	0.0114[2]	0.0325[5]	0.0127[3]	0.0155[4]	0.0785[10]	0.0532[7]	0.0794[11]
	$\lvert Bias\rvert(\hat{\lambda})$	15.4937[15]	5.6865[5]	7.07[14]	6.1102[6]	6.348[7]	6.4275[8]	6.6041[9]	4.8682[4]	0.0098[1]	6.8481[11]	0.0594[2]	0.0919[3]	6.9848[13]	6.7706[10]	6.8935[12]
	$\lvert Bias\rvert(\hat{c})$	0.095[3]	0.6103[12]	1.0889[15]	0.3703[9]	0.8437[14]	0.6099[11]	0.6518[13]	0.3302[7]	0.099[4]	0.2598[5]	0.0801[1]	0.0894[2]	0.3081[6]	0.5434[10]	0.3489[8]
	$\lvert Bias\rvert(\hat{k})$	5.4108[15]	3.1158[10]	3.6098[13]	2.7712[7]	4.4458[14]	3.3056[12]	3.0923[9]	1.7092[4]	0.0297[1]	2.7691[6]	0.056[2]	0.0772[3]	3.1392[11]	2.4343[5]	2.9933[8]
	$MSE(\hat{\delta})$	0.0103[1]	0.0296[8]	0.0525[15]	0.0317[9]	0.0479[13]	0.0487[14]	0.0458[12]	0.0198[4]	0.0143[2]	0.0208[6]	0.0205[5]	0.0185[3]	0.0433[10]	0.0295[7]	0.0435[11]
	$MSE(\hat{\lambda})$	288.0277[15]	104.9777[5]	128.8454[13]	110.0097[6]	115.3261[7]	118.4189[8]	122.6019[10]	93.4375[4]	0.0626[1]	125.0549[11]	0.1463[2]	0.2843[3]	128.9329[14]	122.2953[9]	127.4316[12]
	$MSE(\hat{c})$	0.2157[3]	1.7266[11]	6.4058[15]	1.3622[8]	3.0336[14]	1.8787[12]	2.1226[13]	1.6052[10]	0.1278[1]	0.7588[5]	0.192[2]	0.3402[4]	1.0789[6]	1.4235[9]	1.1333[7]
10	$MSE(\hat{k})$	52.0169[13]	48.0857[11]	57.4429[14]	36.1286[6]	69.3208[15]	50.3778[12]	47.5978[10]	16.1935[4]	0.0835[1]	37.5423[7]	0.3505[2]	0.3988[3]	44.3491[9]	31.618[5]	43.7566[8]
	$MRE(\hat{\delta})$	0.132[1]	0.2074[9]	0.3058[15]	0.1962[7]	0.2849[13]	0.2947[14]	0.2753[12]	0.1561[4]	0.1398[2]	0.1686[6]	0.1651[5]	0.1493[3]	0.2465[10]	0.2064[8]	0.2483[11]
	$MRE(\hat{\lambda})$	12.4954[15]	5.085[5]	6.3021[14]	5.2491[6]	5.5228[7]	5.6762[8]	5.8421[9]	4.1685[4]	0.1367[1]	5.936[11]	0.1452[2]	0.2734[3]	6.0009[12]	5.9037[10]	6.0131[13]
	$MRE(\hat{c})$	0.3217[2]	0.9769[10]	1.515[15]	0.755[7]	1.256[14]	1.0377[12]	1.067[13]	0.7268[6]	0.2958[1]	0.6976[5]	0.3417[3]	0.4123[4]	0.861[8]	0.9796[11]	0.8811[9]
	$MRE(\hat{k})$	3.6696[15]	2.6172[10]	3.0725[13]	2.1365[5]	3.4528[14]	2.7488[12]	2.6687[11]	1.3934[4]	0.14[1]	2.2881[7]	0.1914[2]	0.2534[3]	2.4779[9]	2.153[6]	2.4685[8]
	D_{abs}	0.0828[3]	0.0868[5]	0.0898[12]	0.0881[7]	0.0889[9]	0.0886[8]	0.0875[6]	0.0896[11]	0.0821[2]	0.0854[4]	0.075[1]	0.0914[13]	0.0941[15]	0.0892[10]	0.0936[14]
	D_{max}	0.1515[4]	0.1547[7]	0.1656[14]	0.1559[9]	0.1662[15]	0.157[10]	0.1552[8]	0.1574[11]	0.1414[2]	0.1464[3]	0.1341[1]	0.1525[5]	0.1641[13]	0.1525[6]	0.1632[12]
	$\sum$ Ranks	113[8]	117[9]	196[15]	98[6]	168[14]	156[13]	148[12]	78[4]	22[1]	92[5]	33[2]	56[3]	146[11]	113[8]	144[10]
	$\lvert Bias\rvert(\hat{\delta})$	0.0174[6]	0.0138[5]	0.0239[11]	0.0289[12]	0.0305[13]	0.0441[15]	0.0369[14]	0.0067[3]	0.0078[4]	0.0222[10]	$2e-04$[1]	$6e-04$[2]	0.0174[8]	0.0196[9]	0.0174[7]
	$\lvert Bias\rvert(\hat{\lambda})$	9.8817[15]	5.7698[10]	5.8337[11]	5.3499[9]	7.4154[14]	5.1933[8]	4.2146[6]	2.3605[4]	0.0264[2]	4.7279[7]	0.0187[1]	0.0383[3]	5.9522[12]	3.9457[5]	6.0805[13]
	$\lvert Bias\rvert(\hat{c})$	0.0984[5]	0.2593[8]	0.5152[11]	0.4191[9]	0.5669[14]	0.6653[15]	0.554[13]	0.1867[7]	0.0407[2]	0.4631[10]	0.0252[1]	0.0441[3]	0.1092[6]	0.5302[12]	0.0898[4]
	$\lvert Bias\rvert(\hat{k})$	2.6454[15]	1.5075[10]	1.4497[9]	1.5574[11]	2.1659[14]	1.2562[8]	0.9777[7]	0.8361[4]	0.0524[3]	0.9125[5]	0.0016[1]	0.0166[2]	1.5924[12]	0.9768[6]	1.6916[13]
	$MSE(\hat{\delta})$	$8e-04$[3]	0.0041[6]	0.0049[8]	0.013[15]	0.0106[12]	0.0112[14]	0.0106[13]	0.005[10]	0.0018[4]	0.0046[7]	$5e-04$[1]	$6e-04$[2]	0.0051[11]	0.0037[5]	0.005[9]
	$MSE(\hat{\lambda})$	183.6125[15]	108.0428[10]	109.613[11]	100.8497[9]	141.8314[14]	90.5871[8]	72.4535[6]	45.6056[4]	0.0336[2]	89.4359[7]	0.0235[1]	0.0451[3]	110.7891[13]	70.5039[5]	110.758[12]
	$MSE(\hat{c})$	0.0243[2]	0.5003[7]	1.2745[10]	1.1938[9]	1.6095[13]	2.0078[15]	1.5415[11]	0.6111[8]	0.031[4]	1.545[12]	0.0228[1]	0.0256[3]	0.4869[5]	1.6664[14]	0.4875[6]
20	$MSE(\hat{k})$	14.8978[12]	11.2157[9]	12.6248[10]	14.3177[11]	17.3739[15]	9.9615[8]	7.5184[6]	9.7455[7]	0.0639[2]	6.9212[4]	0.0407[1]	0.082[3]	17.1008[13]	7.3464[5]	17.103[14]
	$MRE(\hat{\delta})$	0.0347[3]	0.0677[6]	0.0922[11]	0.099[12]	0.1053[13]	0.1167[15]	0.1068[14]	0.0552[4]	0.0566[5]	0.0791[10]	0.031[1]	0.0329[2]	0.0712[7]	0.0736[9]	0.0717[8]
	$MRE(\hat{\lambda})$	8.2722[15]	5.2267[10]	5.4193[12]	4.8326[9]	6.5649[14]	4.8192[8]	4.09[6]	2.1862[4]	0.1145[2]	4.4555[7]	0.0702[1]	0.1204[3]	5.4073[11]	3.7395[5]	5.4294[13]
	$MRE(\hat{c})$	0.1695[4]	0.5154[8]	0.8195[10]	0.7217[9]	0.8854[14]	1.0381[15]	0.8701[13]	0.4239[5]	0.1592[3]	0.8439[11]	0.119[1]	0.1434[2]	0.4728[6]	0.8671[12]	0.4886[7]
	$MRE(\hat{k})$	1.9268[14]	1.4858[9]	1.6197[13]	1.5067[10]	2.0302[15]	1.4701[8]	1.3015[7]	0.8088[4]	0.1318[2]	1.1907[5]	0.0809[1]	0.1383[3]	1.5561[11]	1.2031[6]	1.5653[12]
	D_{abs}	0.058[2]	0.0619[6]	0.0653[10]	0.0659[11]	0.0664[12]	0.0642[9]	0.0626[8]	0.0697[13]	0.0626[7]	0.0601[4]	0.0413[1]	0.0583[3]	0.073[15]	0.0606[5]	0.0724[14]
	D_{max}	0.0985[3]	0.108[7]	0.1196[13]	0.1144[9]	0.1193[12]	0.1156[10]	0.1111[8]	0.1174[11]	0.1056[5]	0.1044[4]	0.0744[1]	0.0959[2]	0.1326[15]	0.1061[6]	0.1313[14]
	$\sum$ Ranks	114[8]	111[7]	150[13]	145[10]	189[15]	156[14]	132[9]	88[4]	47[3]	103[5]	14[1]	36[2]	145[10]	104[6]	146[12]
	$\lvert Bias\rvert(\hat{\delta})$	0.0083[8]	0.0104[9]	0.0242[12]	0.0082[7]	0.0337[14]	0.0401[15]	0.0195[10]	0.0035[5]	0.0016[2]	0.0072[6]	0.0013[1]	0.0028[3]	0.0268[13]	0.0032[4]	0.0214[11]
	$\lvert Bias\rvert(\hat{\lambda})$	4.3078[15]	1.9732[8]	2.529[12]	1.7214[6]	2.8282[14]	2.1696[10]	2.1093[9]	1.9492[7]	0.0043[1]	1.2267[5]	0.016[2]	0.0404[3]	2.3341[11]	0.874[4]	2.5399[13]
	$\lvert Bias\rvert(\hat{c})$	0.0201[2]	0.2681[9]	0.3453[11]	0.2438[8]	0.5273[14]	0.5372[15]	0.4304[13]	0.023[3]	0.0499[5]	0.2867[10]	0.0262[4]	0.002[1]	0.3529[12]	0.0787[6]	0.2286[7]
	$\lvert Bias\rvert(\hat{k})$	1.2146[15]	0.2945[9]	0.3411[11]	0.4088[12]	0.4303[13]	0.1875[6]	0.2666[8]	0.5534[14]	0.0044[1]	0.0816[5]	0.0324[3]	0.0056[2]	0.2431[7]	0.0442[4]	0.3253[10]
	$MSE(\hat{\delta})$	$2e-04$[5]	0.0017[9]	0.0027[11]	0.0014[7]	0.0076[15]	0.0076[14]	0.0027[10]	$1e-04$[4]	$3e-04$[6]	0.0016[8]	$1e-04$[3]	$1e-04$[1]	0.0049[13]	$1e-04$[2]	0.0033[12]
	$MSE(\hat{\lambda})$	72.5126[15]	39.9198[9]	47.8463[13]	30.5452[6]	52.2971[14]	39.3942[8]	40.9254[10]	34.501[7]	0.0092[2]	20.1062[5]	0.0032[1]	0.0202[3]	45.3934[11]	16.8003[4]	46.6583[12]
	$MSE(\hat{c})$	0.0119[4]	0.6523[9]	0.4437[7]	0.6544[10]	1.6539[15]	1.4139[13]	1.6006[14]	0.019[5]	0.0112[3]	1.0973[11]	0.0106[2]	0.0097[1]	1.2665[12]	0.0505[6]	0.5525[8]
50	$MSE(\hat{k})$	6.1258[15]	3.1528[11]	3.8089[13]	2.9462[7]	3.9889[14]	3.0718[10]	3.2502[12]	2.2356[6]	0.0158[2]	1.4064[5]	0.0108[1]	0.0257[3]	2.9947[8]	1.0696[4]	3.0548[9]
	$MRE(\hat{\delta})$	0.0166[5]	0.0396[9]	0.0567[12]	0.0376[8]	0.0842[14]	0.0934[15]	0.0507[10]	0.0161[4]	0.0239[6]	0.0307[7]	0.015[3]	0.0142[1]	0.0676[13]	0.0147[2]	0.0564[11]
	$MRE(\hat{\lambda})$	3.8126[15]	2.1196[8]	2.7917[13]	1.757[7]	2.9154[14]	2.4668[10]	2.3469[9]	1.6171[6]	0.0602[2]	1.5629[5]	0.0248[1]	0.0815[3]	2.5225[11]	1.2179[4]	2.5921[12]
	$MRE(\hat{c})$	0.1132[4]	0.4276[8]	0.521[11]	0.401[7]	0.7691[14]	0.7854[15]	0.6447[12]	0.1251[5]	0.1049[3]	0.4765[9]	0.0783[1]	0.0888[2]	0.6531[13]	0.184[6]	0.507[10]
	$MRE(\hat{k})$	1.0392[15]	0.6557[8]	0.8892[14]	0.5986[7]	0.8827[13]	0.7944[12]	0.7493[11]	0.4071[4]	0.0665[2]	0.5469[6]	0.0359[1]	0.0684[3]	0.7442[10]	0.4183[5]	0.739[9]
	D_{abs}	0.0304[2]	0.0329[4]	0.0336[8]	0.0343[10]	0.0344[11]	0.0339[9]	0.0329[5]	0.0354[13]	0.0326[3]	0.033[6]	0.0203[1]	0.0332[7]	0.0423[14]	0.0352[12]	0.043[15]
	D_{max}	0.0536[2]	0.0584[6]	0.0628[11]	0.0605[9]	0.0639[13]	0.0632[12]	0.059[7]	0.0621[10]	0.0571[4]	0.0575[5]	0.0385[1]	0.0562[3]	0.0754[14]	0.0601[8]	0.0761[15]
	$\sum$ Ranks	122[9]	116[8]	159[12]	111[7]	192[15]	164[14]	140[10]	93[6]	42[3]	93[6]	25[1]	36[2]	162[13]	71[4]	154[11]
	$\lvert Bias\rvert(\hat{\delta})$	0.0046[12]	0.0011[2]	0.0245[15]	0.0024[4]	0.014[13]	0.018[14]	$1e-04$[1]	0.0037[11]	0.0033[8]	0.002[3]	0.0025[5]	0.0025[6]	0.0033[9]	0.0029[7]	0.0033[10]
	$\lvert Bias\rvert(\hat{\lambda})$	1.1813[9]	1.4987[10]	2.0641[13]	1.5875[12]	3.0303[15]	2.8949[14]	1.5592[11]	0.0253[2]	0.0254[3]	0.0586[4]	0.0185[1]	0.078[5]	0.4782[7]	0.2002[6]	0.4968[8]
	$\lvert Bias\rvert(\hat{c})$	0.0108[1]	0.053[12]	0.1622[15]	0.0474[9]	0.1192[13]	0.133[14]	0.0392[6]	0.0406[7]	0.021[4]	0.0374[5]	0.0116[3]	0.0111[2]	0.0493[10]	0.052[11]	0.0474[8]
	$\lvert Bias\rvert(\hat{k})$	0.3265[12]	0.245[9]	0.2764[11]	0.446[14]	0.5177[15]	0.4252[13]	0.275[10]	0.0753[7]	0.0373[3]	0.1226[8]	0.0359[1]	0.0363[2]	0.0548[5]	0.0663[6]	0.041[4]
	$MSE(\hat{\delta})$	0[6]	$4e-04$[12]	0.006[15]	0[7]	0.0027[13]	0.0035[14]	$1e-04$[8]	0[5]	$2e-04$[11]	0[2]	0[1]	0[3]	$2e-04$[9]	0[4]	$2e-04$[10]
	$MSE(\hat{\lambda})$	6.8809[7]	24.5351[12]	31.0911[13]	23.6417[10]	52.1911[15]	51.1189[14]	24.5199[11]	0.6827[4]	0.0085[2]	0.9202[5]	0.0028[1]	0.0655[3]	11.6201[9]	1.6047[6]	11.6074[8]
	$MSE(\hat{c})$	0.0108[5]	0.031[10]	0.2087[15]	0.0208[8]	0.1049[13]	0.1278[14]	0.0265[9]	0.0089[4]	0.0081[3]	0.0168[6]	0.0039[1]	0.0055[2]	0.0529[11]	0.0197[7]	0.0534[12]
100	$MSE(\hat{k})$	0.9039[9]	1.6186[10]	2.0096[12]	3.6906[15]	2.952[14]	2.7085[13]	1.6198[11]	0.268[5]	0.015[2]	0.2183[4]	0.0086[1]	0.0224[3]	0.8786[7]	0.3312[6]	0.8831[8]
	$MRE(\hat{\delta})$	0.0091[6]	0.0154[11]	0.0567[15]	0.0097[7]	0.0377[13]	0.0441[14]	0.0126[8]	0.0077[3]	0.0177[12]	0.0071[1]	0.0072[2]	0.0083[5]	0.0145[10]	0.0081[4]	0.0145[9]
	$MRE(\hat{\lambda})$	1.3103[9]	1.6606[10]	2.2089[13]	1.7705[12]	2.9919[15]	2.9061[14]	1.6834[11]	0.4453[5]	0.0609[2]	0.4447[4]	0.0247[1]	0.1157[3]	0.7737[8]	0.5719[6]	0.7596[7]
	$MRE(\hat{c})$	0.1069[5]	0.1671[10]	0.317[15]	0.1371[8]	0.2829[13]	0.3061[14]	0.1614[9]	0.0869[3]	0.0887[4]	0.1136[6]	0.0549[1]	0.076[2]	0.1986[11]	0.1191[7]	0.1996[12]
	$MRE(\hat{k})$	0.4621[9]	0.5004[10]	0.6285[12]	0.677[13]	0.8028[15]	0.7701[14]	0.5022[11]	0.252[5]	0.0685[3]	0.2194[4]	0.0347[1]	0.0554[2]	0.3147[8]	0.2545[6]	0.3068[7]
	D_{abs}	0.0286[4]	0.0296[9]	0.0303[11]	0.0305[12]	0.031[14]	0.0308[13]	0.0294[8]	0.0293[7]	0.0298[10]	0.028[2]	0.0174[1]	0.0284[3]	0.0292[6]	0.0325[15]	0.029[5]
	D_{max}	0.0506[4]	0.0534[5]	0.0566[13]	0.0539[9]	0.057[15]	0.0568[14]	0.0537[8]	0.0535[7]	0.0534[6]	0.0486[3]	0.0298[1]	0.0477[2]	0.0549[11]	0.0549[12]	0.0546[10]
	$\sum$ Ranks	98[6]	132[11]	188[13]	140[12]	196[15]	193[14]	122[10]	75[5]	73[4]	57[3]	21[1]	43[2]	121[9]	103[7]	118[8]
	$\lvert Bias\rvert(\hat{\delta})$	0.0019[11]	$8e-04$[4]	0.0029[14]	$6e-04$[3]	0.0023[12]	0.003[15]	$9e-04$[5]	0.0017[10]	0.0024[13]	0.0015[6]	0.0016[7]	0.0016[8]	$2e-04$[1]	0.0017[9]	$2e-04$[2]
	$\lvert Bias\rvert(\hat{\lambda})$	0.3446[11]	0.1732[6]	0.3673[13]	0.0615[4]	0.5017[15]	0.4855[14]	0.3514[12]	0.2695[10]	$2e-04$[1]	0.1789[7]	0.0531[3]	0.0345[2]	0.2347[9]	0.1305[5]	0.2278[8]
	$\lvert Bias\rvert(\hat{c})$	0.0165[5]	0.0327[10]	0.0326[9]	0.0523[15]	0.0476[14]	0.0455[12]	0.0225[8]	0.046[13]	0.0211[7]	0.0138[4]	0.0053[2]	0.0165[6]	0.0095[3]	0.0345[11]	$1e-04$[1]
	$\lvert Bias\rvert(\hat{k})$	0.1206[14]	$3e-04$[1]	0.0734[11]	0.136[15]	0.0816[13]	0.0679[10]	0.075[12]	0.0566[9]	0.0029[3]	0.0237[5]	0.0292[6]	0.0449[8]	0.01[4]	0.001[2]	0.0414[7]
	$MSE(\hat{\delta})$	0[5]	0[9]	$1e-04$[13]	0[11]	$1e-04$[14]	$1e-04$[15]	0[7]	0[3]	0[12]	0[6]	0[1]	0[2]	0[10]	0[4]	0[8]
	$MSE(\hat{\lambda})$	1.173[8]	0.8649[7]	1.692[12]	0.3439[3]	2.5024[14]	2.3607[13]	1.4161[11]	2.7814[15]	0.0043[1]	0.7356[6]	0.6287[4]	0.1321[2]	1.3952[10]	0.6462[5]	1.3202[9]
	$MSE(\hat{c})$	0.0077[4]	0.0126[8]	0.0131[11]	0.0193[13]	0.0229[14]	0.0241[15]	0.0127[9]	0.0125[7]	0.0031[2]	0.0083[5]	0.0031[1]	0.0033[3]	0.0151[12]	0.0128[10]	0.0121[6]
200	$MSE(\hat{k})$	0.2433[10]	0.1858[6]	0.3082[12]	0.1568[4]	0.4481[14]	0.4326[13]	0.2674[11]	1.053[15]	0.0093[1]	0.1753[5]	0.14[3]	0.0485[2]	0.2135[9]	0.1873[7]	0.1936[8]
	$MRE(\hat{\delta})$	0.0038[5]	0.0047[8]	0.0108[14]	0.005[10]	0.0097[13]	0.0112[15]	0.0042[7]	0.0037[2]	0.0078[12]	0.0038[6]	0.0034[1]	0.0037[3]	0.0051[11]	0.0037[4]	0.0049[9]
	$MRE(\hat{\lambda})$	0.5197[10]	0.4844[9]	0.6715[12]	0.3697[4]	0.842[14]	0.8456[15]	0.6196[11]	0.8028[13]	0.0442[1]	0.4067[6]	0.2428[3]	0.1918[2]	0.4433[7]	0.4488[8]	0.3775[5]
	$MRE(\hat{c})$	0.0893[5]	0.1123[8]	0.1174[10]	0.125[13]	0.1532[14]	0.1583[15]	0.1179[11]	0.1056[7]	0.0502[1]	0.0887[4]	0.0571[3]	0.0533[2]	0.1225[12]	0.0976[6]	0.1143[9]
	$MRE(\hat{k})$	0.2277[10]	0.2158[8]	0.2833[12]	0.1965[7]	0.3366[13]	0.3441[14]	0.2602[11]	0.4482[15]	0.0545[1]	0.1866[6]	0.1327[3]	0.1099[2]	0.1829[5]	0.2201[9]	0.1649[4]
	D_{abs}	0.0208[5]	0.0206[2]	0.0206[1]	0.0214[8]	0.0211[7]	0.0211[6]	0.0208[4]	0.0229[13]	0.022[10]	0.0207[3]	0.0221[11]	0.0217[9]	0.0244[15]	0.0224[12]	0.0241[14]
	D_{max}	0.0353[1]	0.0356[2]	0.0374[8]	0.037[7]	0.0383[11]	0.0384[12]	0.0361[4]	0.0419[13]	0.0365[6]	0.0357[3]	0.038[10]	0.0364[5]	0.0443[15]	0.0378[9]	0.0421[14]
	$\sum$ Ranks	104[8]	88[5]	152[13]	117[9]	182[14]	184[15]	123[10]	145[12]	71[3]	72[4]	58[2]	56[1]	123[10]	101[6]	104[8]
	$\lvert Bias\rvert(\hat{\delta})$	0.0016[10]	$3e-04$[3]	0.0046[13]	$1e-04$[1]	0.0054[14]	0.006[15]	$5e-04$[4]	0.0014[7]	$2e-04$[2]	0.0016[10]	0.0015[8]	0.0016[10]	$9e-04$[5]	0.0016[10]	0.001[6]
	$\lvert Bias\rvert(\hat{\lambda})$	0.0474[4]	0.1119[9]	0.0254[3]	0.4964[14]	0.3662[13]	0.3419[12]	0.0679[7]	0.5108[15]	0.0129[1]	0.066[6]	0.0984[8]	0.1132[10]	0.1586[11]	0.0505[5]	0.0199[2]
	$\lvert Bias\rvert(\hat{c})$	0.0394[7]	0.0596[11]	0.0729[13]	0.0626[12]	0.0998[14]	0.1005[15]	0.0527[9]	0.056[10]	0.0273[3]	0.0367[6]	0.0403[8]	0.0359[5]	0.0101[2]	0.0358[4]	0.0058[1]
	$\lvert Bias\rvert(\hat{k})$	0.0365[6]	0.0611[8]	0.1117[13]	0.0417[7]	0.0198[4]	0.0044[1]	0.0795[9]	0.1176[14]	0.017[3]	0.1087[11]	0.1112[12]	0.1493[15]	0.0304[5]	0.0977[10]	0.0055[2]
	$MSE(\hat{\delta})$	0[8]	0[12]	$1e-04$[13]	0[11]	$4e-04$[14]	$5e-04$[15]	0[10]	0[4]	0[1]	0[8]	0[5]	0[8]	0[2]	0[8]	0[3]
	$MSE(\hat{\lambda})$	0.4117[7]	1.1785[11]	0.4599[8]	4.9401[14]	2.0904[13]	1.964[12]	1.1082[10]	7.5345[15]	0.001[1]	0.278[5]	0.0645[3]	0.2055[4]	0.5033[9]	0.3437[6]	0.0041[2]
	$MSE(\hat{c})$	0.0042[6]	0.0125[11]	0.0162[13]	0.0142[12]	0.0602[14]	0.0649[15]	0.0085[9]	0.0086[10]	0.0019[1]	0.0038[4]	0.0039[5]	0.0034[3]	0.0047[8]	0.0044[7]	0.0026[2]
300	$MSE(\hat{k})$	0.1355[8]	0.2912[11]	0.1894[9]	0.7738[14]	0.5382[13]	0.5173[12]	0.2528[10]	2.0548[15]	0.0021[1]	0.1111[6]	0.0444[3]	0.1021[4]	0.1049[5]	0.1329[7]	0.0055[2]
	$MRE(\hat{\delta})$	0.0032[2]	0.0059[12]	0.0147[13]	0.005[11]	0.0159[14]	0.017[15]	0.0043[10]	0.0033[7]	0.0034[9]	0.0032[2]	0.0033[6]	0.0032[2]	0.0034[8]	0.0032[2]	0.0032[5]
	$MRE(\hat{\lambda})$	0.3191[7]	0.5119[11]	0.3769[9]	0.8597[14]	0.7888[13]	0.774[12]	0.4529[10]	1.0021[15]	0.0205[1]	0.2945[6]	0.1069[3]	0.2384[5]	0.2121[4]	0.3432[8]	0.0326[2]
	$MRE(\hat{c})$	0.0702[8]	0.1047[12]	0.1032[11]	0.1204[13]	0.1674[14]	0.1692[15]	0.0876[9]	0.096[10]	0.038[1]	0.0664[5]	0.0621[3]	0.0638[4]	0.0684[7]	0.0672[6]	0.0476[2]
	$MRE(\hat{k})$	0.1837[7]	0.2713[11]	0.2293[9]	0.3672[12]	0.3837[14]	0.3791[13]	0.2327[10]	0.5008[15]	0.0249[1]	0.1804[6]	0.0783[3]	0.1571[5]	0.1037[4]	0.1977[8]	0.0367[2]
	D_{abs}	0.0133[3]	0.0141[7]	0.0137[5]	0.0148[10]	0.0148[9]	0.0147[8]	0.014[6]	0.0153[11]	0.012[1]	0.0128[2]	0.0164[14]	0.0166[15]	0.0163[13]	0.0136[4]	0.0153[12]
	D_{max}	0.023[3]	0.0242[6]	0.0271[11]	0.0255[7]	0.0276[12]	0.0276[13]	0.0234[4]	0.03[15]	0.0204[1]	0.0221[2]	0.027[10]	0.0267[9]	0.0288[14]	0.0238[5]	0.0259[8]
	$\sum$ Ranks	86[4]	135[10]	143[11]	152[12]	175[15]	173[14]	117[9]	163[13]	27[1]	79[3]	91[6]	99[8]	97[7]	90[5]	51[2]

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Table 6: Partial and overall ranks for all estimation methods of the SHBXII model from Table 5.

n	Est.	MLE	ADE	RTADE	LTADE	CVME	OLSE	WLSE	ADSOE	KE	MPSE	MSADE	MSALDE	MSSDE	MSSLDE	MSLNDE
	$\lvert Bias\rvert(\hat{\delta})$	8	9	14	6	12	15	13	1	2	5	3	4	10	7	11
	$\lvert Bias\rvert(\hat{\lambda})$	15	5	14	6	7	8	9	4	1	11	2	3	13	10	12
	$\lvert Bias\rvert(\hat{c})$	3	12	15	9	14	11	13	7	4	5	1	2	6	10	8
	$\lvert Bias\rvert(\hat{k})$	15	10	13	7	14	12	9	4	1	6	2	3	11	5	8
	$MSE(\hat{\delta})$	1	8	15	9	13	14	12	4	2	6	5	3	10	7	11
	$MSE(\hat{\lambda})$	15	5	13	6	7	8	10	4	1	11	2	3	14	9	12
	$MSE(\hat{c})$	3	11	15	8	14	12	13	10	1	5	2	4	6	9	7
10	$MSE(\hat{k})$	13	11	14	6	15	12	10	4	1	7	2	3	9	5	8
	$MRE(\hat{\delta})$	1	9	15	7	13	14	12	4	2	6	5	3	10	8	11
	$MRE(\hat{\lambda})$	15	5	14	6	7	8	9	4	1	11	2	3	12	10	13
	$MRE(\hat{c})$	2	10	15	7	14	12	13	6	1	5	3	4	8	11	9
	$MRE(\hat{k})$	15	10	13	5	14	12	11	4	1	7	2	3	9	6	8
	D_{abs}	3	5	12	7	9	8	6	11	2	4	1	13	15	10	14
	D_{max}	4	7	14	9	15	10	8	11	2	3	1	5	13	6	12
	$\sum$ Ranks	$113^{\{8\}}$	$117^{\{9\}}$	$196^{\{15\}}$	$98^{\{6\}}$	$168^{\{14\}}$	$156^{\{13\}}$	$148^{\{12\}}$	$78^{\{4\}}$	$22^{\{1\}}$	$92^{\{5\}}$	$33^{\{2\}}$	$56^{\{3\}}$	$146^{\{11\}}$	$113^{\{8\}}$	$144^{\{10\}}$
	$\lvert Bias\rvert(\hat{\delta})$	6	5	11	12	13	15	14	3	4	10	1	2	8	9	7
	$\lvert Bias\rvert(\hat{\lambda})$	15	10	11	9	14	8	6	4	2	7	1	3	12	5	13
	$\lvert Bias\rvert(\hat{c})$	5	8	11	9	14	15	13	7	2	10	1	3	6	12	4
	$\lvert Bias\rvert(\hat{k})$	15	10	9	11	14	8	7	4	3	5	1	2	12	6	13
	$MSE(\hat{\delta})$	3	6	8	15	12	14	13	10	4	7	1	2	11	5	9
	$MSE(\hat{\lambda})$	15	10	11	9	14	8	6	4	2	7	1	3	13	5	12
	$MSE(\hat{c})$	2	7	10	9	13	15	11	8	4	12	1	3	5	14	6
20	$MSE(\hat{k})$	12	9	10	11	15	8	6	7	2	4	1	3	13	5	14
	$MRE(\hat{\delta})$	3	6	11	12	13	15	14	4	5	10	1	2	7	9	8
	$MRE(\hat{\lambda})$	15	10	12	9	14	8	6	4	2	7	1	3	11	5	13
	$MRE(\hat{c})$	4	8	10	9	14	15	13	5	3	11	1	2	6	12	7
	$MRE(\hat{k})$	14	9	13	10	15	8	7	4	2	5	1	3	11	6	12
	D_{abs}	2	6	10	11	12	9	8	13	7	4	1	3	15	5	14
	D_{max}	3	7	13	9	12	10	8	11	5	4	1	2	15	6	14
	$\sum$ Ranks	$114^{\{8\}}$	$111^{\{7\}}$	$150^{\{13\}}$	$145^{\{10\}}$	$189^{\{15\}}$	$156^{\{14\}}$	$132^{\{9\}}$	$88^{\{4\}}$	$47^{\{3\}}$	$103^{\{5\}}$	$14^{\{1\}}$	$36^{\{2\}}$	$145^{\{10\}}$	$104^{\{6\}}$	$146^{\{12\}}$
	$\lvert Bias\rvert(\hat{\delta})$	8	9	12	7	14	15	10	5	2	6	1	3	13	4	11
	$\lvert Bias\rvert(\hat{\lambda})$	15	8	12	6	14	10	9	7	1	5	2	3	11	4	13
	$\lvert Bias\rvert(\hat{c})$	2	9	11	8	14	15	13	3	5	10	4	1	12	6	7
	$\lvert Bias\rvert(\hat{k})$	15	9	11	12	13	6	8	14	1	5	3	2	7	4	10
	$MSE(\hat{\delta})$	5	9	11	7	15	14	10	4	6	8	3	1	13	2	12
	$MSE(\hat{\lambda})$	15	9	13	6	14	8	10	7	2	5	1	3	11	4	12
	$MSE(\hat{c})$	4	9	7	10	15	13	14	5	3	11	2	1	12	6	8
50	$MSE(\hat{k})$	15	11	13	7	14	10	12	6	2	5	1	3	8	4	9
	$MRE(\hat{\delta})$	5	9	12	8	14	15	10	4	6	7	3	1	13	2	11
	$MRE(\hat{\lambda})$	15	8	13	7	14	10	9	6	2	5	1	3	11	4	12
	$MRE(\hat{c})$	4	8	11	7	14	15	12	5	3	9	1	2	13	6	10
	$MRE(\hat{k})$	15	8	14	7	13	12	11	4	2	6	1	3	10	5	9
	D_{abs}	2	4	8	10	11	9	5	13	3	6	1	7	14	12	15
	D_{max}	2	6	11	9	13	12	7	10	4	5	1	3	14	8	15
	$\sum$ Ranks	$122^{\{9\}}$	$116^{\{8\}}$	$159^{\{12\}}$	$111^{\{7\}}$	$192^{\{15\}}$	$164^{\{14\}}$	$140^{\{10\}}$	$93^{\{6\}}$	$42^{\{3\}}$	$93^{\{6\}}$	$25^{\{1\}}$	$36^{\{2\}}$	$162^{\{13\}}$	$71^{\{4\}}$	$154^{\{11\}}$
	$\lvert Bias\rvert(\hat{\delta})$	12	2	15	4	13	14	1	11	8	3	5	6	9	7	10
	$\lvert Bias\rvert(\hat{\lambda})$	9	10	13	12	15	14	11	2	3	4	1	5	7	6	8
	$\lvert Bias\rvert(\hat{c})$	1	12	15	9	13	14	6	7	4	5	3	2	10	11	8
	$\lvert Bias\rvert(\hat{k})$	12	9	11	14	15	13	10	7	3	8	1	2	5	6	4
	$MSE(\hat{\delta})$	6	12	15	7	13	14	8	5	11	2	1	3	9	4	10
	$MSE(\hat{\lambda})$	7	12	13	10	15	14	11	4	2	5	1	3	9	6	8
	$MSE(\hat{c})$	5	10	15	8	13	14	9	4	3	6	1	2	11	7	12
100	$MSE(\hat{k})$	9	10	12	15	14	13	11	5	2	4	1	3	7	6	8
	$MRE(\hat{\delta})$	6	11	15	7	13	14	8	3	12	1	2	5	10	4	9
	$MRE(\hat{\lambda})$	9	10	13	12	15	14	11	5	2	4	1	3	8	6	7
	$MRE(\hat{c})$	5	10	15	8	13	14	9	3	4	6	1	2	11	7	12
	$MRE(\hat{k})$	9	10	12	13	15	14	11	5	3	4	1	2	8	6	7
	D_{abs}	4	9	11	12	14	13	8	7	10	2	1	3	6	15	5
	D_{max}	4	5	13	9	15	14	8	7	6	3	1	2	11	12	10
	$\sum$ Ranks	$98^{\{6\}}$	$132^{\{11\}}$	$188^{\{13\}}$	$140^{\{12\}}$	$196^{\{15\}}$	$193^{\{14\}}$	$122^{\{10\}}$	$75^{\{5\}}$	$73^{\{4\}}$	$57^{\{3\}}$	$21^{\{1\}}$	$43^{\{2\}}$	$121^{\{9\}}$	$103^{\{7\}}$	$118^{\{8\}}$
	$\lvert Bias\rvert(\hat{\delta})$	11	4	14	3	12	15	5	10	13	6	7	8	1	9	2
	$\lvert Bias\rvert(\hat{\lambda})$	11	6	13	4	15	14	12	10	1	7	3	2	9	5	8
	$\lvert Bias\rvert(\hat{c})$	5	10	9	15	14	12	8	13	7	4	2	6	3	11	1
	$\lvert Bias\rvert(\hat{k})$	14	1	11	15	13	10	12	9	3	5	6	8	4	2	7
	$MSE(\hat{\delta})$	5	9	13	11	14	15	7	3	12	6	1	2	10	4	8
	$MSE(\hat{\lambda})$	8	7	12	3	14	13	11	15	1	6	4	2	10	5	9
	$MSE(\hat{c})$	4	8	11	13	14	15	9	7	2	5	1	3	12	10	6
200	$MSE(\hat{k})$	10	6	12	4	14	13	11	15	1	5	3	2	9	7	8
	$MRE(\hat{\delta})$	5	8	14	10	13	15	7	2	12	6	1	3	11	4	9
	$MRE(\hat{\lambda})$	10	9	12	4	14	15	11	13	1	6	3	2	7	8	5
	$MRE(\hat{c})$	5	8	10	13	14	15	11	7	1	4	3	2	12	6	9
	$MRE(\hat{k})$	10	8	12	7	13	14	11	15	1	6	3	2	5	9	4
	D_{abs}	5	2	1	8	7	6	4	13	10	3	11	9	15	12	14
	D_{max}	1	2	8	7	11	12	4	13	6	3	10	5	15	9	14
	$\sum$ Ranks	$104^{\{8\}}$	$88^{\{5\}}$	$152^{\{13\}}$	$117^{\{9\}}$	$182^{\{14\}}$	$184^{\{15\}}$	$123^{\{10\}}$	$145^{\{12\}}$	$71^{\{3\}}$	$72^{\{4\}}$	$58^{\{2\}}$	$56^{\{1\}}$	$123^{\{10\}}$	$101^{\{6\}}$	$104^{\{8\}}$
	$\lvert Bias\rvert(\hat{\delta})$	10.5	3	13	1	14	15	4	7	2	10.5	8	10.5	5	10.5	6
	$\lvert Bias\rvert(\hat{\lambda})$	4	9	3	14	13	12	7	15	1	6	8	10	11	5	2
	$\lvert Bias\rvert(\hat{c})$	7	11	13	12	14	15	9	10	3	6	8	5	2	4	1
	$\lvert Bias\rvert(\hat{k})$	6	8	13	7	4	1	9	14	3	11	12	15	5	10	2
	$MSE(\hat{\delta})$	7.5	12	13	11	14	15	10	4	1	7.5	5	7.5	2	7.5	3
	$MSE(\hat{\lambda})$	7	11	8	14	13	12	10	15	1	5	3	4	9	6	2
	$MSE(\hat{c})$	6	11	13	12	14	15	9	10	1	4	5	3	8	7	2
300	$MSE(\hat{k})$	8	11	9	14	13	12	10	15	1	6	3	4	5	7	2
	$MRE(\hat{\delta})$	2.5	12	13	11	14	15	10	7	9	2.5	6	2.5	8	2.5	5
	$MRE(\hat{\lambda})$	7	11	9	14	13	12	10	15	1	6	3	5	4	8	2
	$MRE(\hat{c})$	8	12	11	13	14	15	9	10	1	5	3	4	7	6	2
	$MRE(\hat{k})$	7	11	9	12	14	13	10	15	1	6	3	5	4	8	2
	D_{abs}	3	7	5	10	9	8	6	11	1	2	14	15	13	4	12
	D_{max}	3	6	11	7	12	13	4	15	1	2	10	9	14	5	8
	$\sum$ Ranks	$86^{\{4\}}$	$135^{\{10\}}$	$143^{\{11\}}$	$152^{\{12\}}$	$175^{\{15\}}$	$173^{\{14\}}$	$117^{\{9\}}$	$163^{\{13\}}$	$27^{\{1\}}$	$79^{\{3\}}$	$91^{\{6\}}$	$99^{\{8\}}$	$97^{\{7\}}$	$90^{\{5\}}$	$51^{\{2\}}$

Table 7: Results from simulation Case IV.

n	Est.	MLE	ADE	RTADE	LTADE	CVME	OLSE	WLSE	ADSOE	KE	MPSE	MSADE	MSALDE	MSSDE	MSSLDE	MSLNDE
	$\lvert\text{Bias}\rvert(\hat{\delta})$	$0.0869^{(15)}$	$0.016^{(6)}$	$0.0237^{(11)}$	$0.0032^{(3)}$	$0.0319^{(12)}$	$0.0719^{(14)}$	$0.0518^{(13)}$	$0.0185^{(9)}$	$0.0159^{(5)}$	$0.0127^{(4)}$	$0.0025^{(1)}$	$0.0026^{(2)}$	$0.0169^{(7)}$	$0.0189^{(10)}$	$0.0178^{(8)}$
	$\lvert\text{Bias}\rvert(\hat{\lambda})$	$15.8609^{(15)}$	$4.4718^{(9)}$	$4.9632^{(11)}$	$5.6711^{(14)}$	$4.8853^{(10)}$	$2.9645^{(4)}$	$3.2339^{(5)}$	$3.68^{(6)}$	$0.0083^{(1)}$	$5.1558^{(12)}$	$0.0409^{(2)}$	$0.2562^{(3)}$	$3.8495^{(7)}$	$5.3955^{(13)}$	$4.1715^{(8)}$
	$\lvert\text{Bias}\rvert(\hat{c})$	$0.4449^{(9)}$	$0.4388^{(8)}$	$0.609^{(13)}$	$0.4645^{(10)}$	$0.7396^{(14)}$	$0.7968^{(15)}$	$0.5388^{(12)}$	$0.3402^{(7)}$	$0.0187^{(1)}$	$0.3031^{(6)}$	$0.0668^{(2)}$	$0.1151^{(5)}$	$0.0925^{(4)}$	$0.4933^{(11)}$	$0.0885^{(3)}$
	$\lvert\text{Bias}\rvert(\hat{k})$	$6.4109^{(15)}$	$2.2969^{(11)}$	$2.6598^{(13)}$	$2.0638^{(8)}$	$2.8787^{(14)}$	$0.9521^{(4)}$	$1.4596^{(6)}$	$2.1568^{(9)}$	$0.0081^{(2)}$	$2.2274^{(10)}$	$0.0045^{(1)}$	$0.2192^{(3)}$	$1.4039^{(5)}$	$2.6287^{(12)}$	$1.5365^{(7)}$
	$\text{MSE}(\hat{\delta})$	$0.0161^{(8)}$	$0.0185^{(11)}$	$0.0192^{(12)}$	$0.0157^{(7)}$	$0.0194^{(14)}$	$0.0214^{(15)}$	$0.0193^{(13)}$	$0.0175^{(10)}$	$0.0093^{(1)}$	$0.0162^{(9)}$	$0.0103^{(2)}$	$0.0104^{(3)}$	$0.0153^{(4)}$	$0.0157^{(6)}$	$0.0154^{(5)}$
	$\text{MSE}(\hat{\lambda})$	$292.9462^{(15)}$	$80.2564^{(9)}$	$87.56^{(10)}$	$108.7294^{(14)}$	$91.3905^{(11)}$	$59.997^{(4)}$	$62.0699^{(5)}$	$67.8638^{(6)}$	$0.07^{(1)}$	$97.5775^{(13)}$	$0.1138^{(2)}$	$1.5139^{(3)}$	$73.5524^{(7)}$	$97.0128^{(12)}$	$76.0162^{(8)}$
	$\text{MSE}(\hat{c})$	$0.2223^{(4)}$	$1.4108^{(9)}$	$2.4213^{(14)}$	$2.0051^{(12)}$	$2.3499^{(13)}$	$2.4298^{(15)}$	$1.113^{(8)}$	$1.6243^{(11)}$	$0.0779^{(1)}$	$1.0098^{(7)}$	$0.0873^{(2)}$	$0.1226^{(3)}$	$0.5698^{(5)}$	$1.5113^{(10)}$	$0.5744^{(6)}$
10	$\text{MSE}(\hat{k})$	$68.6068^{(15)}$	$30.9426^{(10)}$	$40.5437^{(13)}$	$20.4387^{(6)}$	$42.7849^{(14)}$	$14.8676^{(4)}$	$24.8661^{(8)}$	$29.2448^{(9)}$	$0.0288^{(1)}$	$31.9784^{(11)}$	$0.0911^{(2)}$	$1.541^{(3)}$	$19.7865^{(5)}$	$38.5156^{(12)}$	$20.4818^{(7)}$
	$\text{MRE}(\hat{\delta})$	$0.4392^{(4)}$	$0.5246^{(11)}$	$0.5628^{(14)}$	$0.4619^{(5)}$	$0.5393^{(12)}$	$0.6058^{(15)}$	$0.5595^{(13)}$	$0.4873^{(9)}$	$0.3284^{(1)}$	$0.5038^{(10)}$	$0.3575^{(3)}$	$0.3547^{(2)}$	$0.464^{(6)}$	$0.486^{(8)}$	$0.4642^{(7)}$
	$\text{MRE}(\hat{\lambda})$	$21.1761^{(15)}$	$6.4976^{(9)}$	$7.2128^{(11)}$	$8.0319^{(14)}$	$7.1581^{(10)}$	$4.7513^{(4)}$	$5.0374^{(5)}$	$5.2497^{(6)}$	$0.2738^{(1)}$	$7.4937^{(12)}$	$0.2772^{(2)}$	$0.5684^{(3)}$	$5.8733^{(7)}$	$7.698^{(13)}$	$6.2557^{(8)}$
	$\text{MRE}(\hat{c})$	$0.4147^{(4)}$	$0.7019^{(8)}$	$0.8192^{(13)}$	$0.7792^{(12)}$	$0.8823^{(14)}$	$0.9052^{(15)}$	$0.7029^{(9)}$	$0.7519^{(10)}$	$0.1807^{(1)}$	$0.6438^{(7)}$	$0.2014^{(2)}$	$0.2419^{(3)}$	$0.5789^{(5)}$	$0.7639^{(11)}$	$0.5829^{(6)}$
	$\text{MRE}(\hat{k})$	$6.1124^{(15)}$	$2.7224^{(11)}$	$3.1071^{(13)}$	$2.4111^{(9)}$	$3.3598^{(14)}$	$1.7561^{(4)}$	$2.1617^{(7)}$	$2.3953^{(8)}$	$0.1416^{(1)}$	$2.707^{(10)}$	$0.1756^{(2)}$	$0.3476^{(3)}$	$1.9451^{(5)}$	$3.0693^{(12)}$	$2.068^{(6)}$
	D_{abs}	$0.0921^{(3)}$	$0.0938^{(4)}$	$0.096^{(6)}$	$0.0946^{(5)}$	$0.096^{(7)}$	$0.0982^{(12)}$	$0.0969^{(10)}$	$0.1028^{(13)}$	$0.0965^{(8)}$	$0.0968^{(9)}$	$0.089^{(1)}$	$0.0897^{(2)}$	$0.1114^{(14)}$	$0.098^{(11)}$	$0.1124^{(15)}$
	D_{max}	$0.1632^{(9)}$	$0.1604^{(6)}$	$0.1635^{(10)}$	$0.163^{(8)}$	$0.1679^{(12)}$	$0.1609^{(7)}$	$0.1563^{(4)}$	$0.1792^{(13)}$	$0.1536^{(3)}$	$0.1571^{(5)}$	$0.1442^{(1)}$	$0.1458^{(2)}$	$0.1807^{(14)}$	$0.1653^{(11)}$	$0.1821^{(15)}$
	$\sum$ Ranks	$146^{(12)}$	$122^{(7)}$	$164^{(14)}$	$127^{(10)}$	$171^{(15)}$	$132^{(11)}$	$118^{(6)}$	$126^{(9)}$	$28^{(2)}$	$125^{(8)}$	$25^{(1)}$	$40^{(3)}$	$95^{(4)}$	$152^{(13)}$	$109^{(5)}$
	$\lvert\text{Bias}\rvert(\hat{\delta})$	$0.0469^{(12)}$	$0.0445^{(10)}$	$0.0453^{(11)}$	$0.0299^{(6)}$	$0.0503^{(13)}$	$0.0777^{(15)}$	$0.0687^{(14)}$	$0.011^{(4)}$	$0.0038^{(3)}$	$0.0317^{(7)}$	$0.0022^{(1)}$	$0.0033^{(2)}$	$0.0405^{(9)}$	$0.0255^{(5)}$	$0.0403^{(8)}$
	$\lvert\text{Bias}\rvert(\hat{\lambda})$	$12.263^{(15)}$	$2.8519^{(6)}$	$4.3093^{(12)}$	$3.8629^{(10)}$	$3.0933^{(7)}$	$2.5668^{(5)}$	$2.0971^{(4)}$	$4.2072^{(11)}$	$0.0412^{(1)}$	$3.2737^{(8)}$	$0.1014^{(2)}$	$0.2464^{(3)}$	$6.3004^{(13)}$	$3.3963^{(9)}$	$6.5754^{(14)}$
	$\lvert\text{Bias}\rvert(\hat{c})$	$0.2749^{(4)}$	$0.5271^{(9)}$	$0.9202^{(15)}$	$0.4176^{(8)}$	$0.7389^{(12)}$	$0.8036^{(14)}$	$0.7683^{(13)}$	$0.3693^{(6)}$	$0.0911^{(3)}$	$0.353^{(5)}$	$4e-04^{(1)}$	$0.0054^{(2)}$	$0.612^{(11)}$	$0.3873^{(7)}$	$0.6086^{(10)}$
	$\lvert\text{Bias}\rvert(\hat{k})$	$4.9727^{(15)}$	$0.9532^{(6)}$	$1.3365^{(8)}$	$2.2923^{(14)}$	$1.4371^{(9)}$	$0.9113^{(5)}$	$0.4616^{(4)}$	$2.1164^{(13)}$	$0.0066^{(1)}$	$1.2225^{(7)}$	$0.0226^{(2)}$	$0.1209^{(3)}$	$1.6489^{(11)}$	$1.4704^{(10)}$	$1.6689^{(12)}$
	$\text{MSE}(\hat{\delta})$	$0.0044^{(3)}$	$0.0103^{(9)}$	$0.0108^{(12)}$	$0.0091^{(8)}$	$0.0121^{(13)}$	$0.0154^{(15)}$	$0.0128^{(14)}$	$0.0075^{(5)}$	$0.0044^{(4)}$	$0.0078^{(6)}$	$0.0025^{(1)}$	$0.0029^{(2)}$	$0.0103^{(11)}$	$0.0084^{(7)}$	$0.0103^{(10)}$
	$\text{MSE}(\hat{\lambda})$	$214.4561^{(15)}$	$48.0477^{(6)}$	$78.6965^{(12)}$	$65.2082^{(10)}$	$50.7692^{(7)}$	$44.6361^{(5)}$	$37.8816^{(4)}$	$69.9823^{(11)}$	$0.0254^{(1)}$	$56.8407^{(9)}$	$0.5658^{(2)}$	$2.4399^{(3)}$	$114.9826^{(13)}$	$55.2638^{(8)}$	$122.9837^{(14)}$
	$\text{MSE}(\hat{c})$	$0.3821^{(4)}$	$1.1571^{(7)}$	$4.7977^{(13)}$	$1.2887^{(8)}$	$4.0279^{(12)}$	$3.8116^{(10)}$	$3.8706^{(11)}$	$1.707^{(9)}$	$0.0412^{(1)}$	$1.098^{(5)}$	$0.0529^{(2)}$	$0.089^{(3)}$	$8.0521^{(14)}$	$1.1562^{(6)}$	$8.0696^{(15)}$
20	$\text{MSE}(\hat{k})$	$40.8642^{(15)}$	$7.8022^{(5)}$	$11.184^{(6)}$	$28.6163^{(14)}$	$16.5494^{(12)}$	$12.6683^{(7)}$	$4.3456^{(4)}$	$23.8294^{(13)}$	$0.0114^{(1)}$	$13.6569^{(8)}$	$0.1782^{(2)}$	$0.9723^{(3)}$	$14.2249^{(10)}$	$15.9677^{(11)}$	$14.1165^{(9)}$
	$\text{MRE}(\hat{\delta})$	$0.2565^{(4)}$	$0.3899^{(11)}$	$0.3965^{(12)}$	$0.3403^{(8)}$	$0.4235^{(13)}$	$0.5091^{(15)}$	$0.4596^{(14)}$	$0.3099^{(5)}$	$0.2464^{(3)}$	$0.3317^{(6)}$	$0.1752^{(1)}$	$0.1814^{(2)}$	$0.3652^{(10)}$	$0.3394^{(7)}$	$0.3652^{(9)}$
	$\text{MRE}(\hat{\lambda})$	$16.3926^{(15)}$	$4.2035^{(6)}$	$6.2125^{(12)}$	$5.5096^{(10)}$	$4.4965^{(7)}$	$3.9095^{(5)}$	$3.2847^{(4)}$	$5.8304^{(11)}$	$0.1672^{(1)}$	$4.8132^{(8)}$	$0.2861^{(2)}$	$0.5299^{(3)}$	$8.8827^{(13)}$	$4.829^{(9)}$	$9.2344^{(14)}$
	$\text{MRE}(\hat{c})$	$0.3823^{(4)}$	$0.6752^{(9)}$	$1.0503^{(15)}$	$0.5863^{(6)}$	$0.8647^{(10)}$	$0.909^{(12)}$	$0.8756^{(11)}$	$0.6581^{(8)}$	$0.1185^{(1)}$	$0.5909^{(7)}$	$0.1281^{(2)}$	$0.1719^{(3)}$	$0.9735^{(13)}$	$0.5842^{(5)}$	$0.9819^{(14)}$
	$\text{MRE}(\hat{k})$	$4.7785^{(15)}$	$1.4593^{(5)}$	$1.9524^{(10)}$	$2.5887^{(14)}$	$1.8494^{(9)}$	$1.506^{(6)}$	$1.1035^{(4)}$	$2.3382^{(13)}$	$0.081^{(1)}$	$1.7151^{(7)}$	$0.1544^{(2)}$	$0.2443^{(3)}$	$2.1262^{(11)}$	$1.8153^{(8)}$	$2.1291^{(12)}$
	D_{abs}	$0.0543^{(1)}$	$0.0562^{(3)}$	$0.0584^{(8)}$	$0.0592^{(10)}$	$0.0593^{(11)}$	$0.059^{(9)}$	$0.0568^{(4)}$	$0.0625^{(13)}$	$0.057^{(5)}$	$0.0572^{(7)}$	$0.0551^{(2)}$	$0.0571^{(6)}$	$0.0729^{(14)}$	$0.0618^{(12)}$	$0.073^{(15)}$
	D_{max}	$0.0941^{(3)}$	$0.0994^{(7)}$	$0.1091^{(12)}$	$0.1031^{(9)}$	$0.107^{(11)}$	$0.102^{(8)}$	$0.0983^{(6)}$	$0.1093^{(13)}$	$0.0939^{(2)}$	$0.0957^{(5)}$	$0.0876^{(1)}$	$0.0954^{(4)}$	$0.1295^{(14)}$	$0.1038^{(10)}$	$0.1299^{(15)}$
	$\sum$ Ranks	$125^{(8)}$	$99^{(5)}$	$158^{(13)}$	$135^{(10)}$	$146^{(12)}$	$131^{(9)}$	$111^{(6)}$	$135^{(10)}$	$28^{(2)}$	$95^{(4)}$	$23^{(1)}$	$42^{(3)}$	$167^{(14)}$	$114^{(7)}$	$171^{(15)}$
	$\lvert\text{Bias}\rvert(\hat{\delta})$	$0.0177^{(6)}$	$0.0264^{(9)}$	$0.0439^{(14)}$	$0.0192^{(7)}$	$0.036^{(11)}$	$0.0528^{(15)}$	$0.0271^{(10)}$	$0.0163^{(4)}$	$0.002^{(3)}$	$0.0224^{(8)}$	$8e-04^{(1)}$	$0.0014^{(2)}$	$0.0381^{(12)}$	$0.0172^{(5)}$	$0.0383^{(13)}$
	$\lvert\text{Bias}\rvert(\hat{\lambda})$	$3.707^{(15)}$	$0.8315^{(4)}$	$1.0192^{(5)}$	$2.9327^{(12)}$	$2.3113^{(10)}$	$1.761^{(9)}$	$1.4103^{(8)}$	$2.7414^{(11)}$	$0.0197^{(2)}$	$1.0677^{(6)}$	$0.0095^{(1)}$	$0.4238^{(3)}$	$3.1891^{(13)}$	$1.1671^{(7)}$	$3.2152^{(14)}$
	$\lvert\text{Bias}\rvert(\hat{c})$	$0.1173^{(4)}$	$0.387^{(9)}$	$0.4075^{(10)}$	$0.3278^{(6)}$	$0.3629^{(8)}$	$0.4247^{(12)}$	$0.2838^{(5)}$	$0.4423^{(13)}$	$0.0465^{(3)}$	$0.3493^{(7)}$	$4e-04^{(1)}$	$0.0443^{(2)}$	$0.4597^{(14)}$	$0.4202^{(11)}$	$0.4636^{(15)}$
	$\lvert\text{Bias}\rvert(\hat{k})$	$1.8391^{(15)}$	$0.3311^{(4)}$	$0.4078^{(6)}$	$1.141^{(12)}$	$0.8119^{(10)}$	$0.4985^{(9)}$	$0.446^{(8)}$	$1.0412^{(11)}$	$0.0163^{(1)}$	$0.407^{(5)}$	$0.0172^{(2)}$	$0.0409^{(3)}$	$1.4911^{(14)}$	$0.4297^{(7)}$	$1.4907^{(13)}$
	$\text{MSE}(\hat{\delta})$	$0.0012^{(4)}$	$0.0058^{(9)}$	$0.0084^{(14)}$	$0.0053^{(6)}$	$0.007^{(13)}$	$0.0094^{(15)}$	$0.0056^{(8)}$	$0.0059^{(10)}$	$7e-04^{(2)}$	$0.0053^{(7)}$	$7e-04^{(1)}$	$8e-04^{(3)}$	$0.0069^{(11)}$	$0.0047^{(5)}$	$0.0069^{(12)}$
	$\text{MSE}(\hat{\lambda})$	$48.484^{(12)}$	$9.379^{(4)}$	$15.9734^{(6)}$	$47.0154^{(11)}$	$38.6383^{(10)}$	$28.6^{(9)}$	$23.3448^{(8)}$	$48.792^{(13)}$	$0.0088^{(2)}$	$15.4493^{(5)}$	$0.0084^{(1)}$	$7.4424^{(3)}$	$56.4297^{(14)}$	$16.3614^{(7)}$	$57.0069^{(15)}$
	$\text{MSE}(\hat{c})$	$0.1904^{(4)}$	$1.1355^{(9)}$	$0.5856^{(6)}$	$1.2329^{(10)}$	$0.7032^{(7)}$	$0.7667^{(8)}$	$0.5599^{(5)}$	$2.4799^{(15)}$	$0.0213^{(1)}$	$1.2906^{(11)}$	$0.0251^{(2)}$	$0.0642^{(3)}$	$1.8225^{(13)}$	$1.7857^{(12)}$	$1.8548^{(14)}$
50	$\text{MSE}(\hat{k})$	$12.1011^{(13)}$	$3.1072^{(4)}$	$6.2101^{(9)}$	$7.1255^{(11)}$	$6.2433^{(10)}$	$4.2075^{(8)}$	$4.0186^{(7)}$	$9.3133^{(12)}$	$0.0072^{(1)}$	$3.9915^{(6)}$	$0.0073^{(2)}$	$0.2346^{(3)}$	$20.4668^{(15)}$	$3.8385^{(5)}$	$20.4567^{(14)}$
	$\text{MRE}(\hat{\delta})$	$0.1208^{(4)}$	$0.2566^{(10)}$	$0.3075^{(14)}$	$0.2195^{(6)}$	$0.2775^{(11)}$	$0.334^{(15)}$	$0.248^{(9)}$	$0.2301^{(8)}$	$0.0941^{(2)}$	$0.229^{(7)}$	$0.097^{(3)}$	$0.0935^{(1)}$	$0.2801^{(12)}$	$0.2115^{(5)}$	$0.281^{(13)}$
	$\text{MRE}(\hat{\lambda})$	$5.0748^{(15)}$	$1.4934^{(4)}$	$1.8144^{(5)}$	$4.1968^{(12)}$	$3.4856^{(10)}$	$2.8001^{(9)}$	$2.2755^{(8)}$	$3.8505^{(11)}$	$0.0952^{(2)}$	$1.8302^{(6)}$	$0.0837^{(1)}$	$0.6804^{(3)}$	$4.6939^{(13)}$	$1.9705^{(7)}$	$4.7299^{(14)}$
	$\text{MRE}(\hat{c})$	$0.2781^{(4)}$	$0.5335^{(9)}$	$0.5037^{(6)}$	$0.5133^{(8)}$	$0.507^{(7)}$	$0.5542^{(11)}$	$0.4393^{(5)}$	$0.6559^{(13)}$	$0.0976^{(1)}$	$0.5422^{(10)}$	$0.099^{(2)}$	$0.1647^{(3)}$	$0.6829^{(14)}$	$0.5841^{(12)}$	$0.6876^{(15)}$
	$\text{MRE}(\hat{k})$	$1.8809^{(13)}$	$0.802^{(4)}$	$0.9693^{(8)}$	$1.4723^{(12)}$	$1.2953^{(10)}$	$1.0773^{(9)}$	$0.8982^{(7)}$	$1.3145^{(11)}$	$0.0606^{(2)}$	$0.8654^{(5)}$	$0.0599^{(1)}$	$0.1562^{(3)}$	$1.9455^{(14)}$	$0.8872^{(6)}$	$1.9463^{(15)}$
	D_{abs}	$0.0361^{(8)}$	$0.0352^{(2)}$	$0.0355^{(5)}$	$0.0362^{(9)}$	$0.0359^{(6)}$	$0.0361^{(7)}$	$0.0352^{(3)}$	$0.0393^{(11)}$	$0.0343^{(1)}$	$0.0366^{(10)}$	$0.0354^{(4)}$	$0.0414^{(13)}$	$0.0414^{(12)}$	$0.0422^{(15)}$	$0.0415^{(14)}$
	D_{max}	$0.0652^{(6)}$	$0.064^{(4)}$	$0.0659^{(10)}$	$0.0653^{(8)}$	$0.0655^{(9)}$	$0.0645^{(5)}$	$0.0634^{(3)}$	$0.0698^{(11)}$	$0.0593^{(1)}$	$0.0653^{(7)}$	$0.0594^{(2)}$	$0.0701^{(12)}$	$0.0786^{(14)}$	$0.0745^{(13)}$	$0.0787^{(15)}$
	$\sum$ Ranks	$123^{(9)}$	$85^{(4)}$	$118^{(8)}$	$130^{(10)}$	$132^{(11)}$	$141^{(12)}$	$94^{(5)}$	$154^{(13)}$	$24^{(2)}$	$100^{(6)}$	$24^{(2)}$	$57^{(3)}$	$185^{(14)}$	$117^{(7)}$	$196^{(15)}$
	$\lvert\text{Bias}\rvert(\hat{\delta})$	$0.0095^{(10)}$	$0.007^{(8)}$	$0.0346^{(15)}$	$0.0024^{(6)}$	$0.0243^{(13)}$	$0.0313^{(14)}$	$0.0086^{(9)}$	$0.0013^{(2)}$	$0.0053^{(7)}$	$0.0024^{(5)}$	$0.0018^{(4)}$	$7e-04^{(1)}$	$0.0213^{(11)}$	$0.0014^{(3)}$	$0.0214^{(12)}$
	$\lvert\text{Bias}\rvert(\hat{\lambda})$	$1.5546^{(12)}$	$0.6449^{(5)}$	$0.6047^{(4)}$	$1.1759^{(10)}$	$1.2508^{(11)}$	$1.154^{(9)}$	$0.7426^{(6)}$	$1.7103^{(13)}$	$0.011^{(1)}$	$0.8039^{(7)}$	$0.0207^{(2)}$	$0.0696^{(3)}$	$1.8436^{(15)}$	$0.8545^{(8)}$	$1.7385^{(14)}$
	$\lvert\text{Bias}\rvert(\hat{c})$	$0.0957^{(10)}$	$0.0921^{(9)}$	$0.2589^{(15)}$	$0.0551^{(7)}$	$0.1898^{(11)}$	$0.2193^{(14)}$	$0.0887^{(8)}$	$0.031^{(5)}$	$0.0289^{(4)}$	$0.042^{(6)}$	$0.009^{(2)}$	$0.0079^{(1)}$	$0.2159^{(12)}$	$0.0164^{(3)}$	$0.2172^{(13)}$
	$\lvert\text{Bias}\rvert(\hat{k})$	$0.8765^{(15)}$	$0.3112^{(5)}$	$0.2377^{(4)}$	$0.7541^{(13)}$	$0.4887^{(10)}$	$0.3905^{(8)}$	$0.3429^{(6)}$	$0.8163^{(14)}$	$0.0043^{(1)}$	$0.3486^{(7)}$	$0.0266^{(2)}$	$0.0354^{(3)}$	$0.6564^{(12)}$	$0.4652^{(9)}$	$0.6302^{(11)}$
	$\text{MSE}(\hat{\delta})$	$2e-04^{(4)}$	$6e-04^{(10)}$	$0.0044^{(15)}$	$3e-04^{(7)}$	$0.0031^{(13)}$	$0.0038^{(14)}$	$5e-04^{(9)}$	$2e-04^{(6)}$	$4e-04^{(8)}$	$2e-04^{(5)}$	$1e-04^{(2)}$	$1e-04^{(1)}$	$0.0021^{(11)}$	$1e-04^{(3)}$	$0.0021^{(12)}$
	$\text{MSE}(\hat{\lambda})$	$18.9475^{(10)}$	$8.7281^{(5)}$	$8.5891^{(4)}$	$11.1773^{(9)}$	$19.7734^{(12)}$	$19.0631^{(11)}$	$9.4621^{(6)}$	$26.2741^{(14)}$	$0.0051^{(1)}$	$11.0877^{(8)}$	$0.0079^{(2)}$	$0.0647^{(3)}$	$28.6336^{(15)}$	$10.3534^{(7)}$	$25.4586^{(13)}$
	$\text{MSE}(\hat{c})$	$0.0305^{(4)}$	$0.0961^{(8)}$	$0.3112^{(13)}$	$0.1003^{(9)}$	$0.2113^{(11)}$	$0.2445^{(12)}$	$0.0888^{(7)}$	$0.1273^{(10)}$	$0.0098^{(2)}$	$0.0526^{(5)}$	$0.0095^{(1)}$	$0.014^{(3)}$	$0.6275^{(14)}$	$0.0546^{(6)}$	$0.6297^{(15)}$
100	$\text{MSE}(\hat{k})$	$4.6165^{(12)}$	$2.3979^{(4)}$	$2.518^{(6)}$	$4.4673^{(11)}$	$3.9829^{(10)}$	$3.5495^{(9)}$	$2.4959^{(5)}$	$5.6451^{(15)}$	$0.0022^{(1)}$	$2.7253^{(7)}$	$0.0043^{(2)}$	$0.0545^{(3)}$	$5.0238^{(14)}$	$3.0761^{(8)}$	$4.7437^{(13)}$
	$\text{MRE}(\hat{\delta})$	$0.0495^{(6)}$	$0.0728^{(10)}$	$0.2045^{(15)}$	$0.055^{(7)}$	$0.1623^{(13)}$	$0.184^{(14)}$	$0.0726^{(9)}$	$0.049^{(5)}$	$0.0651^{(8)}$	$0.0485^{(4)}$	$0.0375^{(2)}$	$0.036^{(1)}$	$0.1261^{(11)}$	$0.0452^{(3)}$	$0.1264^{(12)}$
	$\text{MRE}(\hat{\lambda})$	$2.205^{(12)}$	$1.186^{(5)}$	$1.1829^{(4)}$	$1.8614^{(9)}$	$2.0217^{(11)}$	$1.9342^{(10)}$	$1.3412^{(6)}$	$2.4296^{(13)}$	$0.0681^{(1)}$	$1.4137^{(8)}$	$0.0849^{(2)}$	$0.1685^{(3)}$	$2.931^{(15)}$	$1.4014^{(7)}$	$2.7912^{(14)}$
	$\text{MRE}(\hat{c})$	$0.1344^{(4)}$	$0.1885^{(9)}$	$0.3349^{(13)}$	$0.1853^{(8)}$	$0.2735^{(11)}$	$0.2963^{(12)}$	$0.1924^{(10)}$	$0.1829^{(7)}$	$0.0561^{(1)}$	$0.1639^{(6)}$	$0.0644^{(2)}$	$0.0834^{(3)}$	$0.3703^{(14)}$	$0.1613^{(5)}$	$0.3706^{(15)}$
	$\text{MRE}(\hat{k})$	$0.9189^{(11)}$	$0.5939^{(4)}$	$0.6499^{(7)}$	$0.9796^{(13)}$	$0.8343^{(10)}$	$0.7954^{(9)}$	$0.641^{(6)}$	$0.9334^{(12)}$	$0.0342^{(1)}$	$0.6342^{(5)}$	$0.0453^{(2)}$	$0.0969^{(3)}$	$1.0973^{(15)}$	$0.6597^{(8)}$	$1.0746^{(14)}$
	D_{abs}	$0.0231^{(2)}$	$0.0236^{(4)}$	$0.0243^{(9)}$	$0.024^{(6)}$	$0.0242^{(8)}$	$0.0241^{(7)}$	$0.0237^{(5)}$	$0.026^{(11)}$	$0.0225^{(1)}$	$0.0234^{(3)}$	$0.0281^{(13)}$	$0.0274^{(12)}$	$0.0297^{(14)}$	$0.0257^{(10)}$	$0.0298^{(15)}$
	D_{max}	$0.0423^{(2)}$	$0.0439^{(4)}$	$0.0454^{(8)}$	$0.0454^{(7)}$	$0.0453^{(6)}$	$0.0455^{(9)}$	$0.0442^{(5)}$	$0.0481^{(13)}$	$0.0372^{(1)}$	$0.043^{(3)}$	$0.0465^{(12)}$	$0.046^{(10)}$	$0.0549^{(14)}$	$0.0464^{(11)}$	$0.055^{(15)}$
	$\sum$ Ranks	$114^{(8)}$	$90^{(5)}$	$132^{(10)}$	$122^{(9)}$	$150^{(12)}$	$152^{(13)}$	$97^{(7)}$	$140^{(11)}$	$38^{(1)}$	$79^{(4)}$	$50^{(2)}$	$50^{(2)}$	$187^{(14)}$	$91^{(6)}$	$188^{(15)}$
	$\lvert\text{Bias}\rvert(\hat{\delta})$	$0.0047^{(10)}$	$0.0036^{(7)}$	$0.0191^{(15)}$	$0.0021^{(6)}$	$0.012^{(13)}$	$0.0152^{(14)}$	$0.0037^{(8)}$	$0.0016^{(5)}$	$0.0045^{(9)}$	$5e-04^{(3)}$	$2e-04^{(2)}$	$2e-04^{(1)}$	$0.0059^{(12)}$	$0.0012^{(4)}$	$0.0058^{(11)}$
	$\lvert\text{Bias}\rvert(\hat{\lambda})$	$0.1614^{(13)}$	$0.0565^{(7)}$	$0.0161^{(6)}$	$0.501^{(14)}$	$0.0953^{(10)}$	$0.0639^{(8)}$	$0.0736^{(9)}$	$0.6688^{(15)}$	$0.0016^{(2)}$	$0.0092^{(5)}$	$0.0013^{(1)}$	$0.0043^{(3)}$	$0.1107^{(11)}$	$0.005^{(4)}$	$0.1198^{(12)}$
	$\lvert\text{Bias}\rvert(\hat{c})$	$0.0585^{(12)}$	$0.0372^{(11)}$	$0.1176^{(15)}$	$0.0189^{(8)}$	$0.0896^{(13)}$	$0.1022^{(14)}$	$0.0308^{(9)}$	$0.0365^{(10)}$	$0.008^{(3)}$	$0.0083^{(4)}$	$0.0099^{(5)}$	$0.0134^{(7)}$	$0.0052^{(2)}$	$0.0119^{(6)}$	$0.0021^{(1)}$
	$\lvert\text{Bias}\rvert(\hat{k})$	$0.1533^{(13)}$	$0.0228^{(6)}$	$0.0689^{(12)}$	$0.264^{(14)}$	$0.0263^{(8)}$	$0.0072^{(3)}$	$0.0309^{(9)}$	$0.4892^{(15)}$	$0.0023^{(1)}$	$0.0234^{(7)}$	$0.0097^{(4)}$	$0.0064^{(2)}$	$0.0352^{(10)}$	$0.0145^{(5)}$	$0.0445^{(11)}$
	$\text{MSE}(\hat{\delta})$	$0^{(6)}$	$2e-04^{(12)}$	$0.0015^{(15)}$	$1e-04^{(8)}$	$7e-04^{(13)}$	$9e-04^{(14)}$	$1e-04^{(9)}$	$0^{(4)}$	$1e-04^{(7)}$	$0^{(5)}$	$0^{(1)}$	$0^{(2)}$	$2e-04^{(10)}$	$0^{(3)}$	$2e-04^{(11)}$
	$\text{MSE}(\hat{\lambda})$	$0.1094^{(7)}$	$0.1233^{(8)}$	$0.0824^{(6)}$	$3.2287^{(14)}$	$0.2433^{(13)}$	$0.2046^{(10)}$	$0.1654^{(9)}$	$9.649^{(15)}$	$0.0025^{(2)}$	$0.0551^{(4)}$	$0.0024^{(1)}$	$0.0107^{(3)}$	$0.2303^{(11)}$	$0.0646^{(5)}$	$0.2322^{(12)}$
	$\text{MSE}(\hat{c})$	$0.0094^{(5)}$	$0.028^{(9)}$	$0.0757^{(15)}$	$0.0325^{(10)}$	$0.0601^{(13)}$	$0.0702^{(14)}$	$0.0249^{(8)}$	$0.0091^{(4)}$	$0.0022^{(1)}$	$0.0129^{(6)}$	$0.003^{(2)}$	$0.0046^{(3)}$	$0.0452^{(11)}$	$0.0149^{(7)}$	$0.0459^{(12)}$
200	$\text{MSE}(\hat{k})$	$0.0846^{(6)}$	$0.1005^{(8)}$	$0.0914^{(7)}$	$0.7687^{(14)}$	$0.1821^{(13)}$	$0.1646^{(12)}$	$0.1143^{(9)}$	$5.0238^{(15)}$	$0.0016^{(1)}$	$0.0515^{(5)}$	$0.0022^{(2)}$	$0.0074^{(3)}$	$0.1467^{(10)}$	$0.047^{(4)}$	$0.1478^{(11)}$
	$\text{MRE}(\hat{\delta})$	$0.0235^{(5)}$	$0.043^{(12)}$	$0.1182^{(15)}$	$0.0382^{(8)}$	$0.0821^{(13)}$	$0.0973^{(14)}$	$0.0394^{(9)}$	$0.0225^{(4)}$	$0.0357^{(7)}$	$0.024^{(6)}$	$0.0191^{(1)}$	$0.0198^{(2)}$	$0.0395^{(11)}$	$0.02^{(3)}$	$0.0394^{(10)}$
	$\text{MRE}(\hat

Table 8: Partial and overall ranks for all estimation methods of the SHBXII model from Table 7.

n	Est.	MLE	ADE	RTADE	LTADE	CVME	OLSE	WLSE	ADSOE	KE	MPSE	MSADE	MSALDE	MSSDE	MSSLDE	MSLNDE
	$\|Bias\|(\hat{\delta})$	15	6	11	3	12	14	13	9	5	4	1	2	7	10	8
	$\|Bias\|(\hat{\lambda})$	15	9	11	14	10	4	5	6	1	12	2	3	7	13	8
	$\|Bias\|(\hat{c})$	9	8	13	10	14	15	12	7	1	6	2	5	4	11	3
	$\|Bias\|(\hat{k})$	15	11	13	8	14	4	6	9	2	10	1	3	5	12	7
	$MSE(\hat{\delta})$	8	11	12	7	14	15	13	10	1	9	2	3	4	6	5
	$MSE(\hat{\lambda})$	15	9	10	14	11	4	5	6	1	13	2	3	7	12	8
	$MSE(\hat{c})$	4	9	14	12	13	15	8	11	1	7	2	3	5	10	6
10	$MSE(\hat{k})$	15	10	13	6	14	4	8	9	1	11	2	3	5	12	7
	$MRE(\hat{\delta})$	4	11	14	5	12	15	13	9	1	10	3	2	6	8	7
	$MRE(\hat{\lambda})$	15	9	11	14	10	4	5	6	1	12	2	3	7	13	8
	$MRE(\hat{c})$	4	8	13	12	14	15	9	10	1	7	2	3	5	11	6
	$MRE(\hat{k})$	15	11	13	9	14	4	7	8	1	10	2	3	5	12	6
	D_{abs}	3	4	6	5	7	12	10	13	8	9	1	2	14	11	15
	D_{max}	9	6	10	8	12	7	4	13	3	5	1	2	14	11	15
	$\sum$ Ranks	$146^{\{12\}}$	$122^{\{7\}}$	$164^{\{14\}}$	$127^{\{10\}}$	$171^{\{15\}}$	$132^{\{11\}}$	$118^{\{6\}}$	$126^{\{9\}}$	$28^{\{2\}}$	$125^{\{8\}}$	$25^{\{1\}}$	$40^{\{3\}}$	$95^{\{4\}}$	$152^{\{13\}}$	$109^{\{5\}}$
	$\|Bias\|(\hat{\delta})$	12	10	11	6	13	15	14	4	3	7	1	2	9	5	8
	$\|Bias\|(\hat{\lambda})$	15	6	12	10	7	5	4	11	1	8	2	3	13	9	14
	$\|Bias\|(\hat{c})$	4	9	15	8	12	14	13	6	3	5	1	2	11	7	10
	$\|Bias\|(\hat{k})$	15	6	8	14	9	5	4	13	1	7	2	3	11	10	12
	$MSE(\hat{\delta})$	3	9	12	8	13	15	14	5	4	6	1	2	11	7	10
	$MSE(\hat{\lambda})$	15	6	12	10	7	5	4	11	1	9	2	3	13	8	14
	$MSE(\hat{c})$	4	7	13	8	12	10	11	9	1	5	2	3	14	6	15
20	$MSE(\hat{k})$	15	5	6	14	12	7	4	13	1	8	2	3	10	11	9
	$MRE(\hat{\delta})$	4	11	12	8	13	15	14	5	3	6	1	2	10	7	9
	$MRE(\hat{\lambda})$	15	6	12	10	7	5	4	11	1	8	2	3	13	9	14
	$MRE(\hat{c})$	4	9	15	6	10	12	11	8	1	7	2	3	13	5	14
	$MRE(\hat{k})$	15	5	10	14	9	6	4	13	1	7	2	3	11	8	12
	D_{abs}	1	3	8	10	11	9	4	13	5	7	2	6	14	12	15
	D_{max}	3	7	12	9	11	8	6	13	2	5	1	4	14	10	15
	$\sum$ Ranks	$125^{\{8\}}$	$99^{\{5\}}$	$158^{\{13\}}$	$135^{\{10\}}$	$146^{\{12\}}$	$131^{\{9\}}$	$111^{\{6\}}$	$135^{\{10\}}$	$28^{\{2\}}$	$95^{\{4\}}$	$23^{\{1\}}$	$42^{\{3\}}$	$167^{\{14\}}$	$114^{\{7\}}$	$171^{\{15\}}$
	$\|Bias\|(\hat{\delta})$	6	9	14	7	11	15	10	4	3	8	1	2	12	5	13
	$\|Bias\|(\hat{\lambda})$	15	4	5	12	10	9	8	11	2	6	1	3	13	7	14
	$\|Bias\|(\hat{c})$	4	9	10	6	8	12	5	13	3	7	1	2	14	11	15
	$\|Bias\|(\hat{k})$	15	4	6	12	10	9	8	11	1	5	2	3	14	7	13
	$MSE(\hat{\delta})$	4	9	14	6	13	15	8	10	2	7	1	3	11	5	12
	$MSE(\hat{\lambda})$	12	4	6	11	10	9	8	13	2	5	1	3	14	7	15
	$MSE(\hat{c})$	4	9	6	10	7	8	5	15	1	11	2	3	13	12	14
50	$MSE(\hat{k})$	13	4	9	11	10	8	7	12	1	6	2	3	15	5	14
	$MRE(\hat{\delta})$	4	10	14	6	11	15	9	8	2	7	3	1	12	5	13
	$MRE(\hat{\lambda})$	15	4	5	12	10	9	8	11	2	6	1	3	13	7	14
	$MRE(\hat{c})$	4	9	6	8	7	11	5	13	1	10	2	3	14	12	15
	$MRE(\hat{k})$	13	4	8	12	10	9	7	11	2	5	1	3	14	6	15
	D_{abs}	8	2	5	9	6	7	3	11	1	10	4	13	12	15	14
	D_{max}	6	4	10	8	9	5	3	11	1	7	2	12	14	13	15
	$\sum$ Ranks	$123^{\{9\}}$	$85^{\{4\}}$	$118^{\{8\}}$	$130^{\{10\}}$	$132^{\{11\}}$	$141^{\{12\}}$	$94^{\{5\}}$	$154^{\{13\}}$	$24^{\{2\}}$	$100^{\{6\}}$	$24^{\{2\}}$	$57^{\{3\}}$	$185^{\{14\}}$	$117^{\{7\}}$	$196^{\{15\}}$
	$\|Bias\|(\hat{\delta})$	10	8	15	6	13	14	9	2	7	5	4	1	11	3	12
	$\|Bias\|(\hat{\lambda})$	12	5	4	10	11	9	6	13	1	7	2	3	15	8	14
	$\|Bias\|(\hat{c})$	10	9	15	7	11	14	8	5	4	6	2	1	12	3	13
	$\|Bias\|(\hat{k})$	15	5	4	13	10	8	6	14	1	7	2	3	12	9	11
	$MSE(\hat{\delta})$	4	10	15	7	13	14	9	6	8	5	2	1	11	3	12
	$MSE(\hat{\lambda})$	10	5	4	9	12	11	6	14	1	8	2	3	15	7	13
	$MSE(\hat{c})$	4	8	13	9	11	12	7	10	2	5	1	3	14	6	15
100	$MSE(\hat{k})$	12	4	6	11	10	9	5	15	1	7	2	3	14	8	13
	$MRE(\hat{\delta})$	6	10	15	7	13	14	9	5	8	4	2	1	11	3	12
	$MRE(\hat{\lambda})$	12	5	4	9	11	10	6	13	1	8	2	3	15	7	14
	$MRE(\hat{c})$	4	9	13	8	11	12	10	7	1	6	2	3	14	5	15
	$MRE(\hat{k})$	11	4	7	13	10	9	6	12	1	5	2	3	15	8	14
	D_{abs}	2	4	9	6	8	7	5	11	1	3	13	12	14	10	15
	D_{max}	2	4	8	7	6	9	5	13	1	3	12	10	14	11	15
	$\sum$ Ranks	$114^{\{8\}}$	$90^{\{5\}}$	$132^{\{10\}}$	$122^{\{9\}}$	$150^{\{12\}}$	$152^{\{13\}}$	$97^{\{7\}}$	$140^{\{11\}}$	$38^{\{1\}}$	$79^{\{4\}}$	$50^{\{2\}}$	$50^{\{2\}}$	$187^{\{14\}}$	$91^{\{6\}}$	$188^{\{15\}}$
	$\|Bias\|(\hat{\delta})$	10	7	15	6	13	14	8	5	9	3	2	1	12	4	11
	$\|Bias\|(\hat{\lambda})$	13	7	6	14	10	8	9	15	2	5	1	3	11	4	12
	$\|Bias\|(\hat{c})$	12	11	15	8	13	14	9	10	3	4	5	7	2	6	1
	$\|Bias\|(\hat{k})$	13	6	12	14	8	3	9	15	1	7	4	2	10	5	11
	$MSE(\hat{\delta})$	6	12	15	8	13	14	9	4	7	5	1	2	10	3	11
	$MSE(\hat{\lambda})$	7	8	6	14	13	10	9	15	2	4	1	3	11	5	12
	$MSE(\hat{c})$	5	9	15	10	13	14	8	4	1	6	2	3	11	7	12
200	$MSE(\hat{k})$	6	8	7	14	13	12	9	15	1	5	2	3	10	4	11
	$MRE(\hat{\delta})$	5	12	15	8	13	14	9	4	7	6	1	2	11	3	10
	$MRE(\hat{\lambda})$	7	8	6	15	12	10	9	14	2	5	1	3	13	4	11
	$MRE(\hat{c})$	5	9	13	11	14	15	8	4	1	6	2	3	10	7	12
	$MRE(\hat{k})$	6	8	7	15	13	12	9	14	1	5	2	3	11	4	10
	D_{abs}	1	6	7	8	10	11	5	9	3	2	4	13	14	12	15
	D_{max}	2	6	7	9	10	11	5	8	1	4	3	13	14	12	15
	$\sum$ Ranks	$98^{\{6\}}$	$117^{\{8\}}$	$146^{\{10\}}$	$154^{\{12\}}$	$168^{\{15\}}$	$162^{\{14\}}$	$115^{\{7\}}$	$136^{\{9\}}$	$41^{\{2\}}$	$67^{\{4\}}$	$31^{\{1\}}$	$61^{\{3\}}$	$150^{\{11\}}$	$80^{\{5\}}$	$154^{\{12\}}$
	$\|Bias\|(\hat{\delta})$	12	2	15	8	13	14	3	11	1	6	7	9	4	10	5
	$\|Bias\|(\hat{\lambda})$	12	7	6	13	15	14	9	10	2	1	3	5	8	4	11
	$\|Bias\|(\hat{c})$	12	10	15	6	13	14	5	8	4	3	7	11	1	9	2
	$\|Bias\|(\hat{k})$	13	9	4	12	15	14	10	11	2	5	1	3	7	6	8
	$MSE(\hat{\delta})$	9	11	15	8	13	14	10	7	12	3	1	2	5	4	6
	$MSE(\hat{\lambda})$	6	7	8	13	15	14	9	12	1	4	2	3	10	5	11
	$MSE(\hat{c})$	4	10	15	7	13	14	9	5	2	6	1	3	11	8	12
300	$MSE(\hat{k})$	6	7	9	13	15	14	8	10	1	4	2	3	11	5	12
	$MRE(\hat{\delta})$	9	11	15	8	13	14	10	5	12	2	1	3	6	4	7
	$MRE(\hat{\lambda})$	7	8	10	13	15	14	9	4	1	5	3	2	12	6	11
	$MRE(\hat{c})$	5	10	15	8	13	14	9	4	3	6	1	2	11	7	12
	$MRE(\hat{k})$	7	8	10	13	15	14	9	4	1	5	2	3	12	6	11
	D_{abs}	1	6	9	8	11	10	5	4	12	2	3	7	14	13	15
	D_{max}	3	8	13	10	11	12	7	2	6	4	1	5	14	9	15
	$\sum$ Ranks	$106^{\{7\}}$	$114^{\{9\}}$	$159^{\{13\}}$	$140^{\{12\}}$	$190^{\{14\}}$	$190^{\{14\}}$	$112^{\{8\}}$	$97^{\{6\}}$	$60^{\{3\}}$	$56^{\{2\}}$	$35^{\{1\}}$	$61^{\{4\}}$	$126^{\{10\}}$	$96^{\{5\}}$	$138^{\{11\}}$

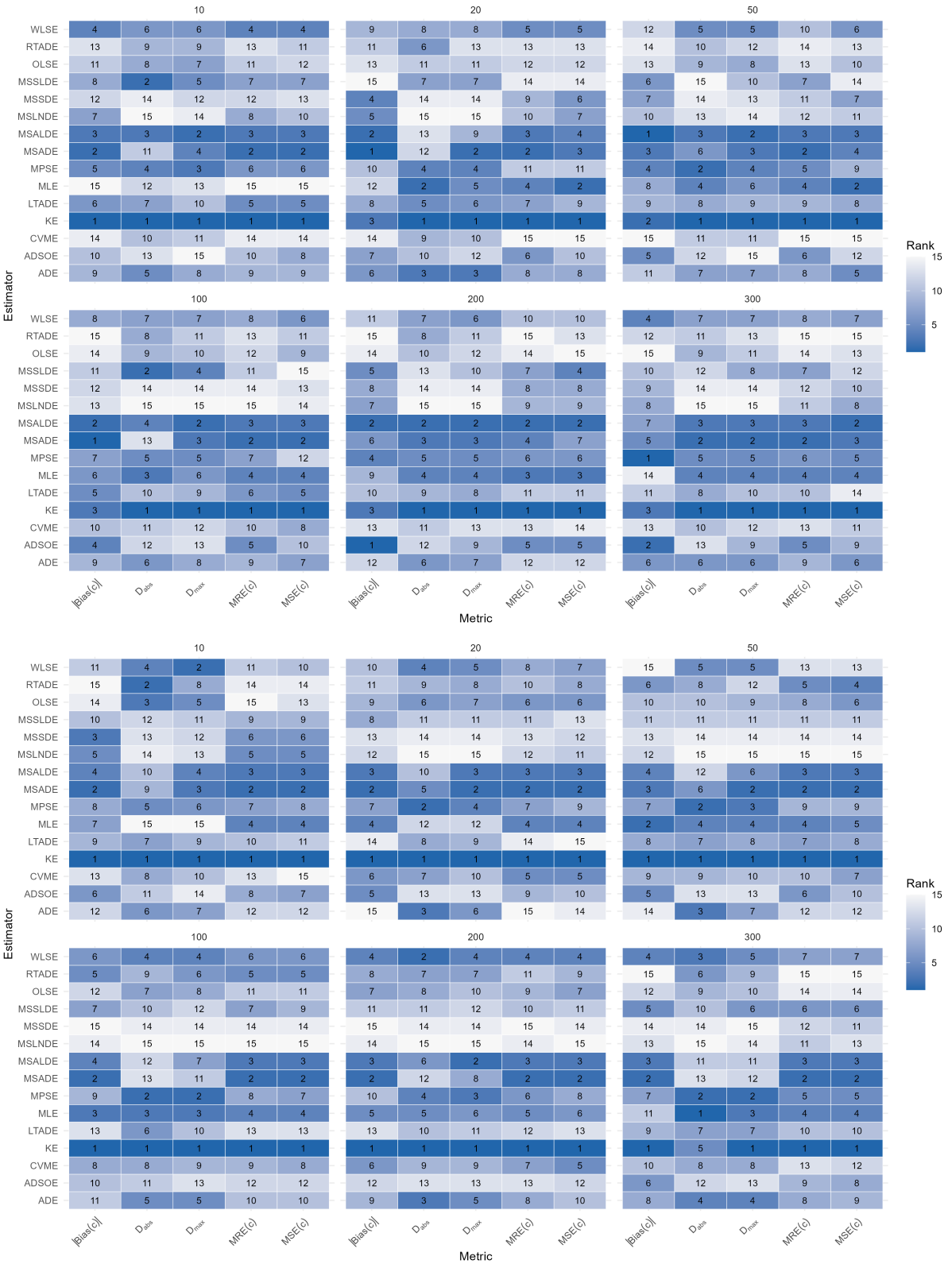


Figure 4: Heatmap simulation Case I and II

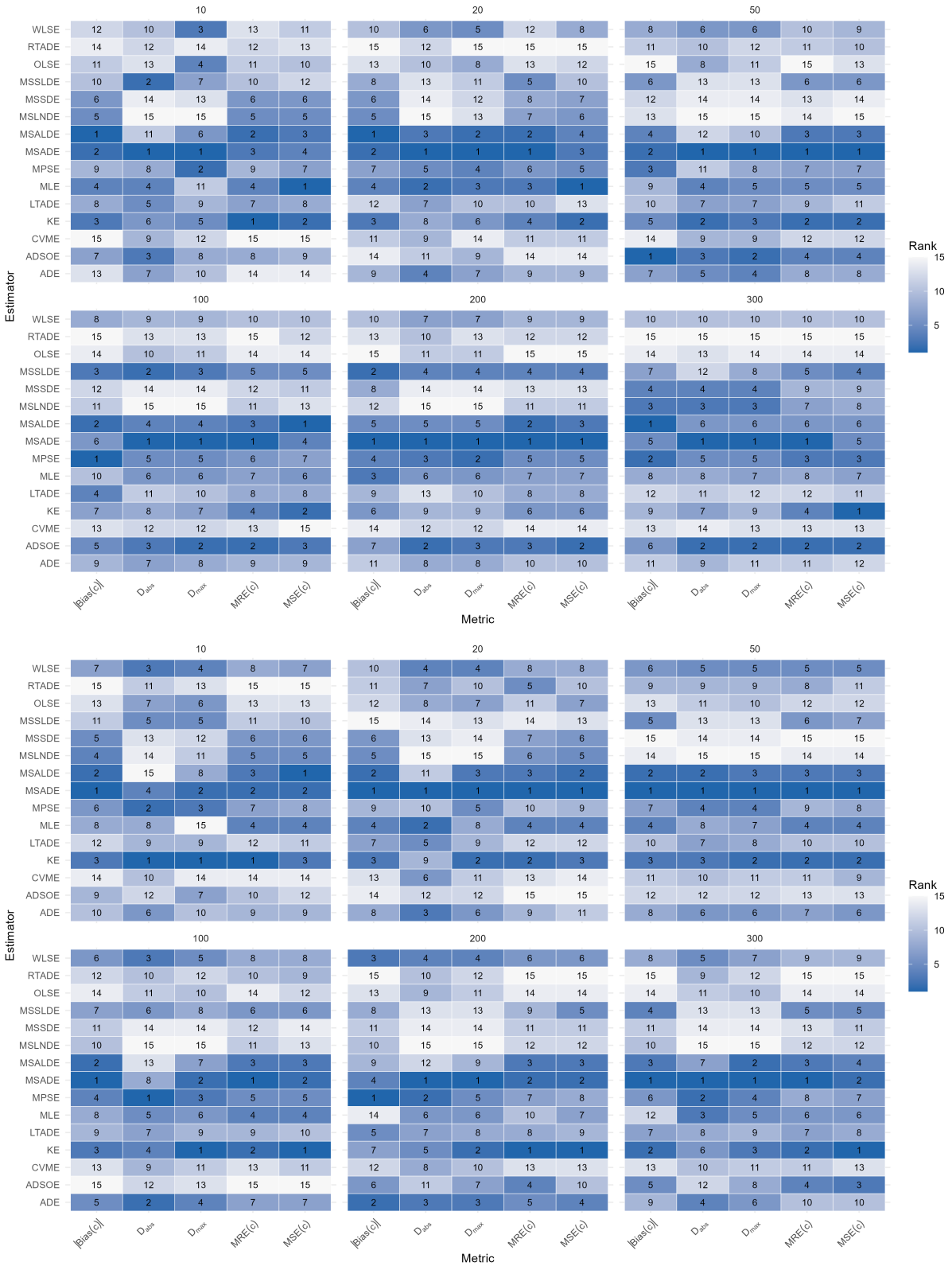


Figure 5: Heatmap simulation Case III and IV

6 Applications

The dataset in Table 9 contains 64 observations of waiting times for patients in the Emergency Department (ED) at NewYork-Presbyterian/Weill Cornell Medical Center in New York. This dataset,

which records the median time patients spend in the ED before being admitted to the hospital, was accessed on June 24, 2024, from the official U.S. Centers for Medicare & Medicaid Services (CMS). The data are made publicly available to promote transparency in healthcare quality.

Table 9: Observed waiting times.

83	51	87	60	28	95	8	27	15	10	18	16	29	54	91	8
17	55	10	35	47	77	36	17	21	36	18	40	10	7	34	27
28	56	8	25	68	146	89	18	73	69	9	37	10	82	29	8
60	61	61	18	169	25	8	26	11	83	11	42	17	14	9	12

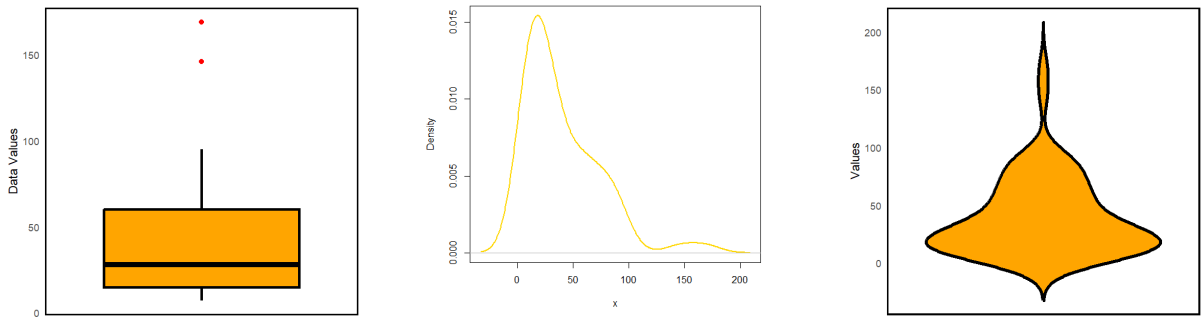
We fit the SHBXII distribution to the observed waiting times dataset in Table 9 and compare it with the standard Burr distribution proposed by Burr [33] and the Weibull distribution proposed by Weibull [48].

Table 10 shows the summary statistics for the observed waiting times dataset. The data consist of 64 observations with a mean of 39.83 and a median of 28, indicating a slightly right-skewed distribution, which is confirmed by the positive skewness of 1.55. The standard deviation (33.75) and variance (1139.10) indicate a moderate spread in the data. The interquartile range of 45.25 reveals a wide middle spread between Q1 = 14.75 and Q3 = 60. Any value that falls outside this range is considered an outlier. The kurtosis value (5.77) suggests a leptokurtic distribution with heavy tails, and extreme values (146, 169) are also present.

Table 10: Summary statistics for observed waiting times.

Statistic	Value
Sample Size (n)	64
Mean ($\bar{x}$)	39.8281
Median	28
Variance (σ^2)	1139.097
Standard Deviation (SD)	33.7505
Range	7–169
First Quartile (Q1)	14.75
Third Quartile (Q3)	60
Interquartile Range (IQR)	45.25
Skewness	1.5464
Kurtosis	5.7711
Outliers	146, 169

Figure 6 contains the boxplot, kernel density plot, and violin plot, which provide nonparametric visualizations of the waiting times data. The visual evidence from these plots strongly indicates that the data are positively skewed, possess a long tail toward higher values, and contain at least two distinct outliers, as shown in the boxplot. The kernel density plot further suggests that the distribution might be multimodal. Table 11 contains the measures of model performance, namely negative log-likelihood (NLL), Akaike information criterion (AIC), consistent AIC (CAIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), Cramér–von Mises (W^*) statistic, Anderson–Darling (A^*) statistic, Kolmogorov–Smirnov (K–S) statistic, and its P-value. The distribution with lower values of the goodness-of-fit statistics and a higher K–S test P-value is considered to provide a better fit to the dataset. The results show that the SHBXII distribution fits the data better than the Weibull and standard Burr distributions commonly used for modeling lifetime data, with a P-value of 0.7428, which is higher than those of the other two distributions. This indicates that the SHBXII distribution provides a good fit to this dataset.



(a) Boxplot (b) kernel density (c) Violin plots

Figure 6: Non-parametric plots of the waiting times data.

Table 11: Model fitness and adequacy measures for the observed waiting time dataset.

Dist.	NLL	AIC	CAIC	BIC	HQIC	W^*	A^*	K-S	P-value
SHBXII	285.4159	578.8318	587.4673	587.4673	582.2337	0.0872	0.6790	0.0851	0.7428
BXII	296.1587	598.3175	604.7941	604.7941	600.8689	0.1193	0.8632	0.1114	0.4053
Weibull	296.9001	597.8003	602.1180	602.1180	599.5012	0.1271	0.9171	0.1113	0.4060

Table 12 presents the MLEs along with their corresponding standard errors (in parentheses) for the parameters of the SHBXII, BXII, and Weibull distributions fitted to the observed waiting time data. The SHBXII is the most heavily parameterized model, utilizing four distinct parameters (c, k, λ, δ), and it demonstrates highly stable and precise estimation as evidenced by its small standard errors, such as $k = 4.4947 \pm 0.05$. In contrast, the standard BXII model exhibits significant parameter instability and estimation uncertainty, particularly for its scale parameter $\lambda = 60.833$ and shape parameter $c = 2.4116$, where the standard errors (63.5079 and 2.7609, respectively) exceed the estimated coefficients. The Weibull distribution provides the most parsimonious fit, relying on only two parameters (c and λ) with relatively stable standard errors.

Table 12: MLE of the parameters for the fitted distributions using waiting time dataset.

Dist.	c	k	λ	δ
SHBXII	0.8309(0.11)	4.4947(0.05)	0.1127(0.05001)	7(1.24)
BXII	2.4116(2.7609)	1.5832(0.3576)	60.833(63.5079)	–
Weibull	1.2744(0.1203)	–	43.2142(4.4915)	–

Figure 7 presents the density, CDF, survival, and TTT plots of the waiting time data. The TTT plot follows the reference line closely, indicating a good fit of the proposed SHBXII distribution to the data. The TTT plot also suggests that the failure rate of the dataset increases over time.

The second dataset in Table 13 contains 62 observations of the mortality rate in Japan from 1960 to 2021. This dataset was accessed on July 21, 2024, from the official World Bank database [49] and was recently studied by Ibeakuzie [50]. The data track the mortality rate of children under five years of age per 1,000 live births and are publicly available to promote transparency in population health research.

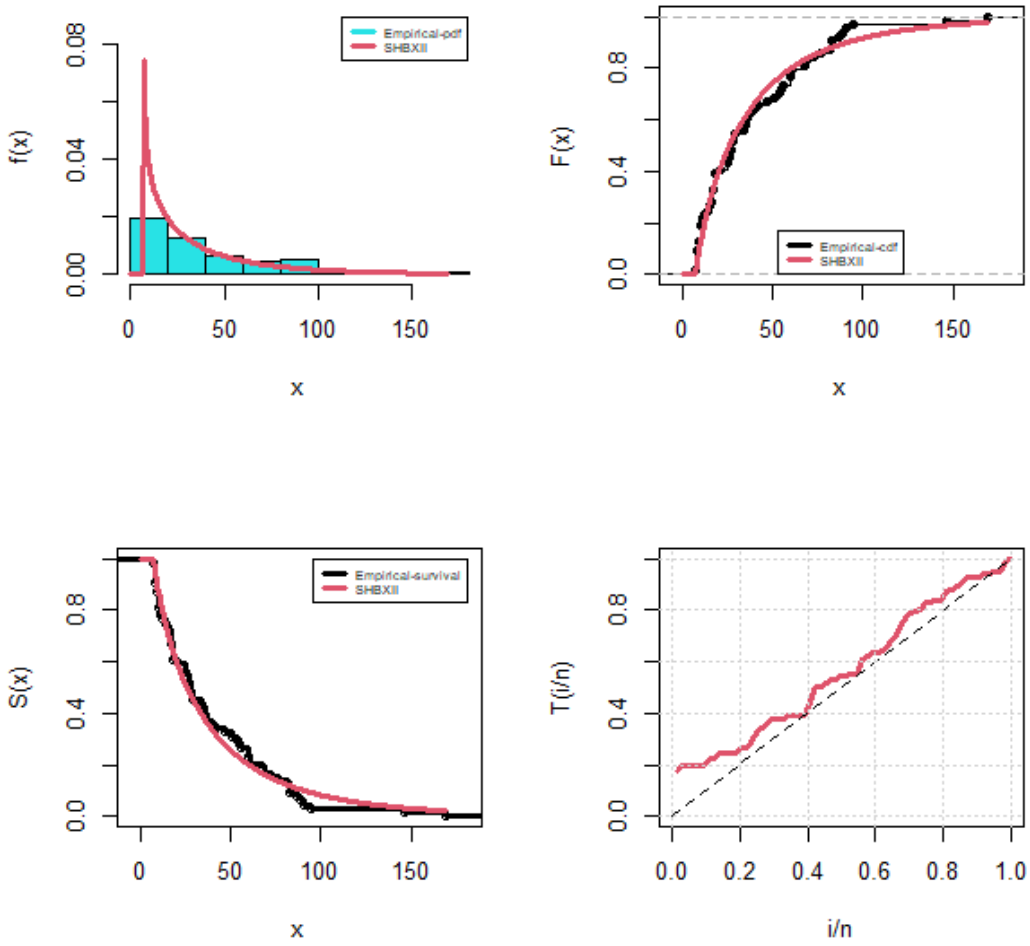


Figure 7: Density, CDF, survival, and TTT plots for waiting time dataset.

Table 13: The mortality rate of children in Japan under 5 years of age, from 1969 to 2021.

39.7	36.2	32.9	29.8	27.0	24.6	22.7	21.0	19.6	18.5	17.5	16.5	15.7	14.8	14.0
13.3	12.5	11.8	11.1	10.5	9.9	9.3	8.8	8.3	7.9	7.5	7.1	6.8	6.6	6.5
6.3	6.2	6.1	6.0	5.9	5.7	5.5	5.2	5.0	4.7	4.5	4.3	4.1	4.0	3.9
3.7	3.6	3.5	3.4	3.3	3.2	3.2	3.0	2.9	2.8	2.8	2.7	2.6	2.5	2.5
2.4	2.3													

Table 14 presents the summary statistics for the dataset consisting of 62 observations. The data have a mean mortality rate of 9.94 and a median of 6.25, indicating a right-skewed distribution, which is supported by the positive skewness value of 1.63. The standard deviation (8.95) and variance (80.16) indicate a moderate level of dispersion around the mean. The interquartile range (9.48) is computed from Q1 = 3.625 and Q3 = 13.1. Values outside this range are considered potential outliers. The kurtosis value of 5.03 indicates a leptokurtic distribution with heavier tails than the normal distribution. The extreme values 39.7, 36.2, 32.9, and 29.8 are considered outliers.

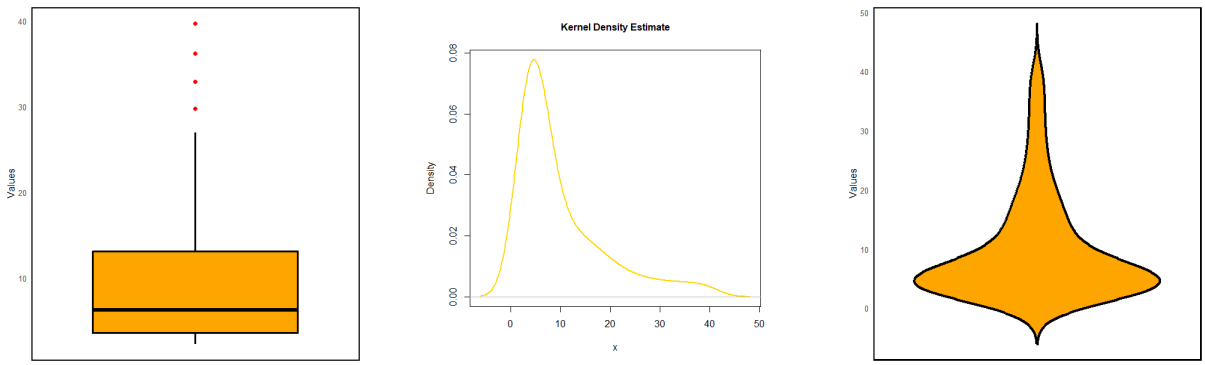
Figure 8 contains the boxplot, kernel density plot, and violin plot, which provide nonparametric visualizations of mortality rate dataset. Again, we assessed the goodness-of-fit of the models using

Table 14: Summary statistics for under 5 mortality rate in Japan.

Statistic	Value
Sample Size (n)	62
Minimum	2.3
Maximum	39.7
Mean ($\bar{x}$)	9.9387
Median	6.25
Variance (σ^2)	80.1595
Standard Deviation (SD)	8.9532
First Quartile (Q1)	3.625
Third Quartile (Q3)	13.1
Interquartile Range (IQR)	9.475
Skewness	1.6319
Kurtosis	5.0264
Outliers	39.7, 36.2, 32.9, 29.8

the same measures discussed in the first data analysis. Based on these criteria found in Table 15, the SHBXII distribution outperforms both the Weibull and the standard Burr distributions, as shown by its highest P-value of 0.9434. This indicates that the SHBXII distribution provides the best fit for the under-5 mortality rate dataset. Table 16 presents the MLEs and their corresponding standard errors (in parentheses) for the parameters of the SHBXII, BXII, and Weibull distributions fitted to the Under 5 mortality dataset. The four-parameter SHBXII model provides stable parameter estimates across the board, with well-bounded standard errors relative to its coefficients ($c = 1.0201 \pm 0.1031$, $k = 1.5717 \pm 0.5681$, $\lambda = 6.3663 \pm 0.10524$, and $\delta = 2.2985 \pm 0.0776$). Conversely, the standard three-parameter BXII model demonstrates acute parameter instability and estimation distress, as seen in its extremely small c estimate (0.0264) and an heavily inflated k estimate (34.3977) that is accompanied by an excessively large standard error (39.2460) exceeding the estimate itself. Meanwhile, the simpler two-parameter Weibull distribution maintains reasonable precision for both its shape parameter $c = 1.2503 \pm 0.1166$ and scale parameter $\lambda = 10.7689 \pm 1.1627$, presenting a much more stable alternative to the standard BXII model for this specific dataset.

Figure 9 shows the density, CDF, survival, and TTT plots of the under 5 mortality rate data. The SHBXII distribution's curve almost perfectly fits the empirical data across all four plots. Additionally, the theoretical SHBXII TTT line in the TTT plot almost perfectly matches the empirical TTT line. This implies that the proposed SHBXII distribution provides a very good fit to the data.



(a) Boxplot (b) kernel density (c) Violin plots

Figure 8: Non-parametric plots of the under 5 mortality rate dataset.

Table 15: Model fitness and adequacy measures for the under 5 mortality dataset.

Dist.	NLL	AIC	CAIC	BIC	HQIC	W^*	A^*	K-S	P-value
SHBXII	-187.8142	383.6284	392.1369	392.1369	386.9690	0.0670	0.0503	0.3620	0.9434
BXII	-192.5180	391.0360	397.4174	397.4174	393.5415	0.1290	0.2422	1.2845	0.2534
Weibull	-201.8367	407.6735	411.9277	411.9277	409.3438	0.1351	0.2846	1.7759	0.2079

Table 16: MLE of the parameters for fitted distributions using under 5 mortality dataset.

Dist.	c	k	λ	δ
SHBXII	1.0201(0.1031)	1.5717(0.5681)	6.3663(0.10524)	2.2985(0.0776)
BXII	0.0264(0.0285)	34.3977(39.2460)	2.3824(0.1291)	–
Weibull	1.2503(0.1166)	–	10.7689(1.1627)	–

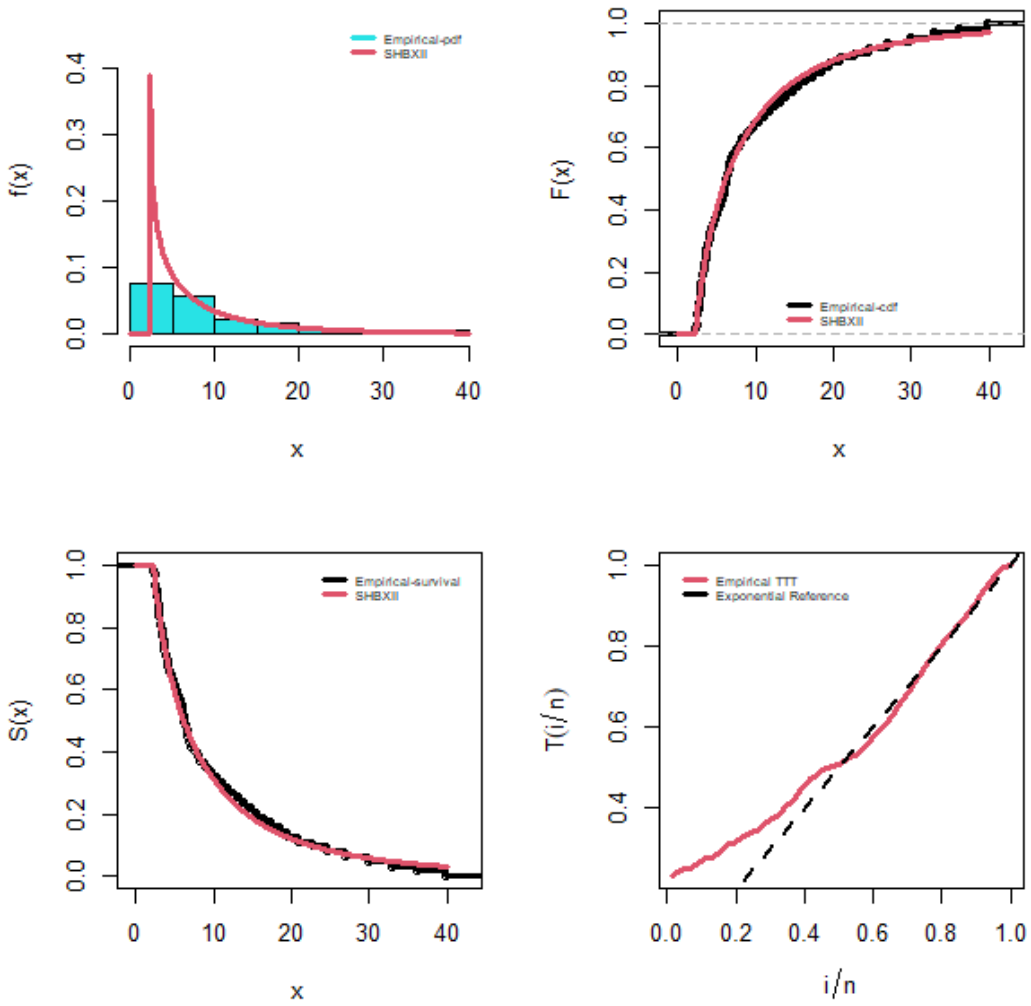


Figure 9: Density, CDF, survival, and TTT plots for under 5 mortality rate dataset.

7 Conclusion

This study has successfully introduced, rigorously investigated, and empirically validated the four-parameter Shifted Burr Type XII distribution as a powerful mathematical generalization of the classical Burr Type XII model. The core contribution of this work lies in formalizing an unknown location parameter ($\delta > 0$) to decouple the distribution's support from its scale, effectively resolving a pervasive real-world boundary limitation in which empirical lifetime or survival observations begin after a non-zero threshold ($y \geq \delta$).

Theoretical derivations confirmed that the Shifted Burr Type XII model exhibits remarkable structural flexibility. Its hazard rate function dynamically accommodates decreasing, unimodal (upside-down bathtub), and heavily skewed profiles, making it uniquely suited for aging and survival analysis. Methodologically, this study evaluated parameter estimation across fifteen distinct non-Bayesian frameworks. An extensive Monte Carlo simulation study demonstrated stable monotonic convergence of the absolute bias and root mean square error toward zero as the sample size increased ($n \to 300$), with Maximum Likelihood Estimation and Maximum Product of Spacings Estimation emerging as the most asymptotically efficient and structurally resilient estimation techniques. Finally, empirical testing on a highly leptokurtic healthcare waiting-time dataset ($n = 64$) showed that the Shifted Burr Type XII model significantly outperforms traditional baseline models, including the classical Burr Type XII and Weibull distributions, yielding lower information-criterion values (Akaike Information Criterion = 578.83, Bayesian Information Criterion = 587.47) and superior Kolmogorov–Smirnov goodness-of-fit results ($D = 0.0851$, p-value = 0.7428).

Despite these substantial contributions, the structural boundary conditions and mathematical properties of the Shifted Burr Type XII distribution reveal key limitations that outline an informative roadmap for future research:

1. **Analytical Moment Constraints and Boundary Stability:** A primary theoretical limitation of the Shifted Burr Type XII model is that its raw and incomplete moments depend heavily on the shape parameters (c and k), which impose strict conditions for numerical convergence. Future work will investigate the exact boundaries of moment existence under extreme parametric variations and develop regularized optimization penalty functions to ensure computational stability near these structural limits.
2. **Adaptation to Complex Censoring Schemes:** The current inferential framework focuses entirely on complete, uncensored data vectors. To maximize the practical value of the Shifted Burr Type XII distribution in industrial reliability engineering and clinical studies, a critical next step is to derive the likelihood and spacing equations under complex censoring mechanisms, including Type-I, Type-II, and progressive censoring configurations.
3. **Bayesian Inference and Informative Priors:** While this study established a robust baseline using fifteen classical frequentist estimators, future research should explore Bayesian estimation paradigms. Implementing Markov Chain Monte Carlo samplers and Gibbs sampling routines, combined with informative and non-informative priors, could provide highly stable and robust parameter estimates, particularly in sparse or small-sample settings.
4. **Multivariate Formulations and Copula Extensions:** This study addressed only the univariate case. Because complex systems frequently exhibit dependent variables operating beyond non-zero thresholds, establishing bivariate or multivariate extensions of the Shifted Burr Type XII distribution via copula functions represents a valuable avenue for risk management, hydrological threshold modeling, and structural reliability analysis.

Funding Statement

The authors declare that they did not receive any external funds for this study.

Acknowledgments

The authors thank anonymous reviewers for their valuable comments.

Data Availability

The data used in this study are available within the manuscript.

Conflict of Interest:

The authors declare no conflicts of interest regarding this study.

Author Contributions

Jude O.: Conceptualization, Methodology, Formal Analysis, Writing–Original Draft Preparation. Okechukwu J. O.: Conceptualization, Validation, Software, Data Curation, Writing–Review & Editing, Supervision, Project Administration. Caleb C. E.: Methodology, Investigation, Resources, Validation. Chidimma U. O.: Formal Analysis, Software, Visualization, Writing–Original Draft Preparation. Chinyere P. O.: Resources, Investigation, Writing–Review & Editing.

AI Usage Statement:

Artificial intelligence tools, namely Gemini AI and Grammarly, were used to assist in language editing, formatting, and improving clarity of expression in this manuscript. The authors reviewed and verified all the generated content to ensure the accuracy and integrity of the scientific information. No AI-based tools were used in the generation, writing, revising, analysis, or interpretation of the research data and findings presented in this article.

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