

Article Type: Research article

A New Extended Rayleigh Model and Its Modelling of Two Engineering Scenarios

Asmaa Abdel-Hakim¹

¹Faculty of Management and International Economy, El Saleheya El Gadida University, New Salhia, Egypt.

✉ Corresponding author
dr_asmaa_abdelhakim@yahoo.com

Article History

Received: 10 November 2025

Revised: 1 December 2025

Accepted: 6 December 2025

Keywords

Log-gamma distribution;
Rayleigh distribution;
Hazard rate function;
Likelihood estimation;
Moments;
Order statistics;
Entropies.

Abstract

This study introduces a new three-parameter log-gamma–Rayleigh (LGR) distribution as a flexible extension of the classical Rayleigh model for lifetime and reliability data. By embedding the Rayleigh distribution within a log-gamma generating framework, the proposed model is capable of capturing a wide spectrum of distributional shapes, including left- and right-skewed patterns, as well as diverse hazard rate forms such as increasing, unimodal, and bathtub-shaped behaviors. A central theoretical contribution is expressing the LGR density as a mixture of Rayleigh densities, which enables tractable derivations of key properties. Explicit expressions are obtained for the quantile function, moments, generating function, and information measures including Rényi and Shannon entropies. Order statistics and their associated densities are also derived through a convergent series approach. Maximum likelihood estimation is employed for parameter inference, supported by closed-form expressions for the observed Fisher information matrix, which yield approximate confidence intervals. The superiority of the LGR distribution is demonstrated through two engineering datasets concerning carbon-fiber breaking stresses and ball-bearing fatigue life. In both applications, the LGR model outperforms classical competitors across multiple goodness-of-fit criteria. The results establish the LGR model as a powerful and versatile tool for modelling reliability and survival data.

1 Introduction

Statistical distributions play a very important role in scientific research, since recognizing the probability distribution of a sample represents a fundamental component in many situations. The gamma distribution is the most popular model for analyzing skewed data. Recently, several new models have been developed based on the gamma distribution. It provides a richness that makes it accurate and appropriate to fit complex datasets. Generalized gamma distributions have been widely studied in statistics and numerous authors have developed various classes of these distributions. Amini et al. [1] proposed two types of general families of continuous distributions called the log-gamma generated (LG-G) families. For any parent cumulative distribution function (CDF) of a random variable with corresponding probability density function (PDF), the two types of LG-G families of distributions are given by

$$F_1(x) = \frac{\Gamma(\alpha, -\beta \log F(x))}{\Gamma(\alpha)}, \quad x \in R, \quad \alpha, \beta > 0, \quad (1)$$

and

$$F_2(x) = \frac{\Gamma(\alpha, -\beta \log \bar{F}(x))}{\Gamma(\alpha)}, \quad x \in R, \quad \alpha, \beta > 0,$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$, $t > 0$, $\alpha > 0$, $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt = \Gamma(\alpha) - \gamma(\alpha, x)$, and $\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt$ are the gamma, upper incomplete gamma, and lower incomplete gamma functions, respectively. The corresponding PDFs of the CDFs of the LG-G families are



$$f_1(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} f(x) (F(x))^{\beta-1} (-\log F(x))^{\alpha-1}, \quad x \in R, \quad \alpha, \beta > 0, \quad (2)$$

and

$$f_2(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} f(x) (\bar{F}(x))^{\beta-1} (-\log \bar{F}(x))^{\alpha-1}, \quad x \in R, \quad \alpha, \beta > 0,$$

respectively.

The Rayleigh distribution is a special case of the Weibull distribution, which provides a population model useful in several areas of statistics, including life testing and reliability studies. This lifetime distribution is of particular interest to engineers and physicists as a model for phenomena involving wave propagation, radiation, and related processes. Recently, extensive studies on estimation, prediction, and other inferential aspects of the Rayleigh distribution have been conducted by several authors, including Ferreira et al. [2], Gupta [3], Kotb and Raqab [4], and Dey et al. [5].

Suppose the random variable X has a Rayleigh distribution, the CDF and PDF of X , respectively are

$$F(x, \lambda) = 1 - e^{-\lambda x^2}, \quad x > 0, \quad \lambda > 0, \quad (3)$$

and

$$f(x, \lambda) = 2\lambda x e^{-\lambda x^2}, \quad x > 0, \quad \lambda > 0, \quad (4)$$

where λ is scale parameter.

Modern reliability and lifetime modelling increasingly relies on flexible distributional forms capable of capturing diverse behaviors in real data, such as skewness, heavy tails, and non-monotone hazard structures. Many studies based on the gamma distribution have been conducted by several authors. For example, Ashour et al. [6] proposed a new lifetime model called the log-gamma–Pareto distribution, derived from log-gamma generated families. Cordeiro et al. [7] introduced the four-parameter gamma extended Weibull distribution, while Cordeiro et al. [8] proposed the gamma–Lomax distribution, a new three-parameter continuous model. Castellares et al. [9] suggested the gamma logistic distribution by introducing an additional skewness parameter as a generalization of the logistic model. More recently, Dias Gonçalves [10] proposed a generalized log-gamma distribution. Classical families, most notably the exponential, Rayleigh, and standard gamma-based models, often fail to accommodate these complexities when applied to contemporary engineering datasets. Motivated by the need for a more adaptable lifetime model that preserves analytical tractability while enhancing modelling flexibility, this study introduces a new three-parameter log-gamma–Rayleigh (LGR) distribution. The proposed model blends the structural advantages of the log-gamma generator with the practical relevance of the Rayleigh baseline, producing a highly flexible distribution capable of representing monotone, unimodal, and bathtub-shaped hazard functions observed in reliability engineering, fatigue studies, and physical systems. Beyond its modelling richness, the LGR family maintains closed-form or series-based expressions for many of its key statistical properties, making it both theoretically appealing and computationally feasible for real applications.

The main contributions of this study can be summarized as follows:

- Introduce a new three-parameter LGR distribution that extends both the gamma–Rayleigh and classical Rayleigh models while retaining interpretability and computational simplicity.
- Establish a series expansion of the LGR density as a mixture of Rayleigh components. This representation enhances theoretical understanding and facilitates the derivation of several structural properties.
- Provide explicit formulations for (i) the quantile function, enabling direct random-variate generation; (ii) the moments and generating function; and (iii) Rényi and Shannon entropies, offering insights into uncertainty and information content.

- Derive closed-form expressions for the PDFs of general order statistics, including the minimum and maximum, using power-series and binomial-expansion techniques.
- Study the maximum likelihood estimators (MLEs) of the model parameters and derive the observed Fisher information matrix, providing practical tools for statistical inference and confidence-interval construction.
- Analyze two engineering datasets, carbon-fiber breaking stresses and ball-bearing fatigue life, and demonstrate that, across multiple goodness-of-fit measures, the LGR distribution outperforms the exponential, Rayleigh, log-Rayleigh, log-gamma, and gamma-Rayleigh models.
- Highlight the LGR model's potential as a robust and flexible alternative for engineering reliability, survival analysis, fatigue modelling, econometric duration data, and other domains requiring adaptable lifetime distributions.

The rest of the paper is organized as follows. In Section 2, we propose a new model called the LGR distribution and provide a series expansion of its density function as a mixture of Rayleigh components. Section 3 presents several mathematical properties, including the quantile function, moments, and generating function. In Section 4, explicit expressions for the Rényi and Shannon entropies are derived. Section 5 develops the densities of the order statistics, including the minimum and maximum. Section 6 examines the maximum likelihood estimation of the model parameters and derives the observed Fisher information matrix. In Section 7, we assess the usefulness of the model using two real datasets. Finally, Section 8 offers concluding remarks.

2 The Log-Gamma Rayleigh Distribution

Inserting (3) and (4) in (1) and (2), the CDF and PDF of the LGR distribution are given by

$$F(x; \alpha, \beta, \lambda) = \frac{\Gamma(\alpha, -\beta \log(1 - e^{-\lambda x^2}))}{\Gamma(\alpha)}, \quad x > 0, \quad \alpha, \beta, \lambda > 0, \quad (5)$$

and

$$f(x; \alpha, \beta, \lambda) = \frac{2\lambda\beta^\alpha x e^{-\lambda x^2}}{\Gamma(\alpha)} (1 - e^{-\lambda x^2})^{\beta-1} \left[-\log(1 - e^{-\lambda x^2}) \right]^{\alpha-1}, \quad x > 0, \quad \alpha, \beta, \lambda > 0, \quad (6)$$

where α and β are shape parameters representing the different patterns of the LGR distribution and λ is scale parameter representing the characteristic life. Hereafter, we denote a random variable X having PDF (5) by $X \sim LGR(\alpha, \beta, \lambda)$. When $\beta = 1$ and $\alpha = \beta = 1$, the LGR distribution reduces to the gamma-Rayleigh and Rayleigh distributions as special cases from (5), respectively. Hence, the LGR distribution can be utilized as a generalization of the gamma-Rayleigh and Rayleigh distributions.

If a random variable Y follows the log-gamma distribution with parameters α and β , then $X = \sqrt{-\lambda^{-1} \log(1 - Y)}$ follows the $LGR(\alpha, \beta, \lambda)$. If a random variable Y follows the gamma distribution with parameters α and β , then $X = \sqrt{-\lambda^{-1} \log(1 - e^{-Y})}$ follows the $LGR(\alpha, \beta, \lambda)$. The survival function, $S(x)$, hazard rate function (HRF), $h(x)$, and cumulative HRF, $H(x)$, of X are given, respectively, as

$$S(x) = 1 - \left\{ \frac{\Gamma(\alpha, -\beta \log(1 - e^{-\lambda x^2}))}{\Gamma(\alpha)} \right\},$$

$$h(x) = \frac{2\lambda\beta^\alpha x e^{-\lambda x^2} (1 - e^{-\lambda x^2})^{\beta-1} \left(-\log(1 - e^{-\lambda x^2}) \right)^{\alpha-1}}{\Gamma(\alpha) - \Gamma(\alpha, -\beta \log(1 - e^{-\lambda x^2}))},$$

and

$$H(x) = -\log \left\{ 1 - \left[\frac{\Gamma(\alpha, -\beta \log(1 - e^{-\lambda x^2}))}{\Gamma(\alpha)} \right] \right\}.$$

Plots of the LGR density for different values of α , β and λ are displayed in Figure 1(a-d). The density plots show left-skewed, right-skewed and symmetrical shapes. As λ grows, depending on the configuration of α , β and λ , the PDF may appear strictly decreasing, moderately right-skewed with a single peak, or sharply unimodal with increasing concentration. Overall, the plots highlight the model's flexibility in representing monotone, skewed, and highly peaked distributions across a wide range of scenarios. Plots of possible shapes for HRF of LGR for different values α , β and λ are displayed in Figure 2(a-d). The model is capable of producing strictly increasing hazard functions, bathtub-shaped patterns with an initial decrease followed by an upward trend, and unimodal hazards that rise to a peak and then decline. These results highlight the flexibility of the distribution in capturing rising, decreasing-increasing, and peaked hazard behaviors commonly encountered in reliability and survival analysis.

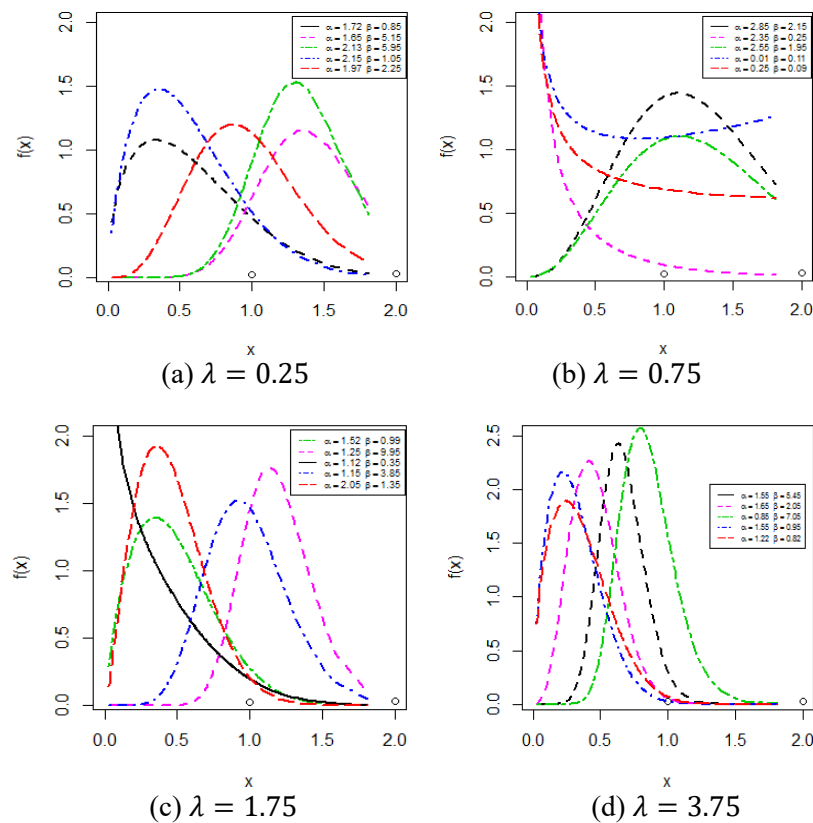


Figure 1: Plots of LGR density with varying parameters.

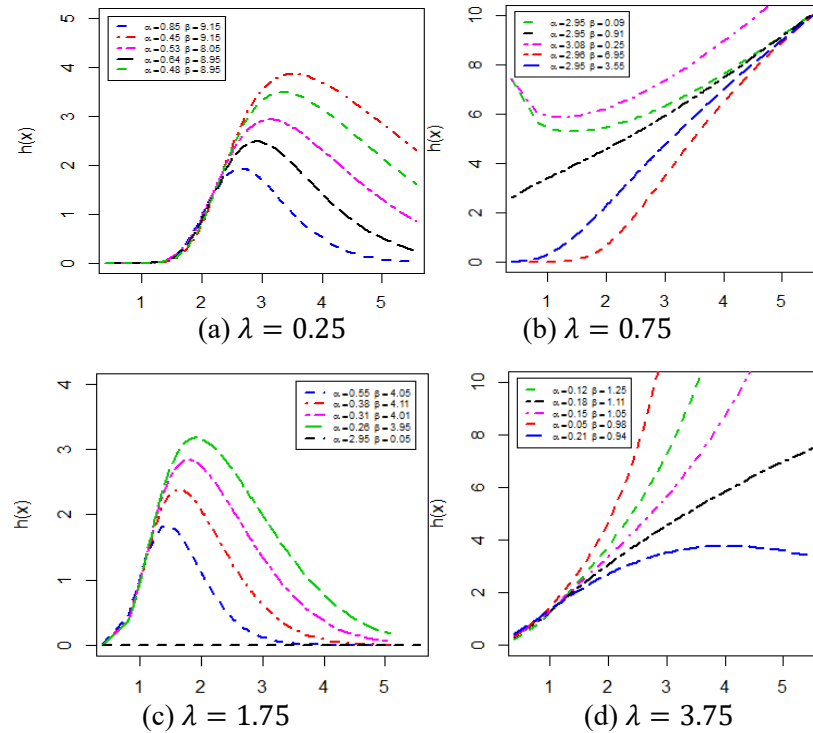


Figure 2: Hazard rate plots for LGR model with varying parameters.

Consider the LGR density in (6) and let $y = e^{-\lambda x^2}$ "for short", (6) can be rewritten as

$$f(x; \alpha, \beta, \lambda) = \frac{2\lambda\beta^\alpha}{\Gamma(\alpha)} xy (1-y)^{\beta-1} [-\log(1-y)]^{\alpha-1}. \quad (7)$$

According to Pinho et al. [11], the power series will be used for the following term

$$-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{(i+1)} = y + \sum_{i=1}^{\infty} \frac{y^{i+1}}{(i+1)},$$

is valid only within its radius of convergence. Specifically, the power series converges absolutely for $|y| < 1$, which ensures that the logarithmic expansion remains finite and well-defined. For real y , this corresponds to the interval $y \in (-1, 1)$; the series diverges at $y = 1$ and converges conditionally at $y = -1$.

Let $i - 1 = s$, hence (7) becomes

$$f(x; \alpha, \beta, \lambda) = \frac{2\lambda\beta^\alpha}{\Gamma(\alpha)} xy (1-y)^{\beta-1} \left[y + \sum_{s=0}^{\infty} \frac{y^{s+2}}{(s+2)} \right]^{\alpha-1}. \quad (8)$$

Using the general binomial expansion, one can write the following terms as

$$\begin{aligned} \left[y + \sum_{s=0}^{\infty} \frac{y^{s+2}}{(s+2)} \right]^{\alpha-1} &= \sum_{j=0}^{\infty} \binom{\alpha-1}{j} y^{\alpha-j-1} \left(\sum_{s=0}^{\infty} \frac{y^{s+2}}{(s+2)} \right)^j \\ &= y^{\alpha-1} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} y^{-j} \left(\sum_{s=0}^{\infty} \frac{y^s}{(s+2)} \right)^j. \end{aligned} \quad (9)$$

and

$$(1-y)^{\beta-1} = \sum_{m=0}^{\infty} (-1)^m \binom{\beta-1}{m} y^m, \quad \beta > 0, \quad (10)$$

is valid only at $|y| < 1$, ensuring that all subsequent derivations involving this expansion remain rigorous, well-defined, and stops at $\beta - 1$ when $\beta > 0$ integer. It should be noted that the expansion in (9) is derived from the power-series representation of the logarithmic function, which converges only for $|y| < 1$. Therefore, the series $\sum_{s=0}^{\infty} (s+2)^{-1} y^{s+2}$ and the resulting binomial-type expansion remain valid exclusively within this convergence region. Imposing the condition $|y| < 1$ ensures absolute convergence of the series and guarantees the correctness of the subsequent analytical developments. Substituting (9) and (10) into (8), then (8) can be rewritten as

$$f(x; \alpha, \beta, \lambda) = \frac{2\lambda\beta^\alpha}{\Gamma(\alpha)} xy^\alpha \left\{ \sum_{m=0}^{\infty} (-1)^m \binom{\beta-1}{m} y^m \right\} \left\{ \sum_{j=0}^{\infty} \binom{\alpha-1}{j} y^j \left(\sum_{s=0}^{\infty} \frac{y^s}{(s+2)} \right)^j \right\}. \quad (11)$$

Using the equation by Gradshteyn and Ryzhik [12] for a power series to a positive integer (j), by setting $a_s = (s+2)^{-1}$, then

$$\left(\sum_{s=0}^{\infty} a_s y^s \right)^j = \sum_{s=0}^{\infty} b_{s,j} y^s, \quad (12)$$

where

$$b_{s,j} = (sa_0)^{-1} \sum_{r=1}^s \{j(r+1)-s\} a_r b_{s-r,j} \quad \text{and} \quad b_{0,j} = a_0^j.$$

Using (12) and replacing y by $e^{-\lambda x^2}$, the density (11) becomes

$$f(x; \alpha, \beta, \lambda) = \frac{2\lambda\beta^\alpha x}{\Gamma(\alpha)} \sum_{m,j,s=0}^{\infty} (-1)^m \binom{\beta-1}{m} \binom{\alpha-1}{j} b_{s,j} e^{-\lambda^* x^2}, \quad (13)$$

where $\lambda^* = \lambda(\alpha + m + j + s)$. Let $f^*(x)$ be the Rayleigh density as in (4), then (13) can be represented as

$$f(x; \alpha, \beta, \lambda) = \sum_{m,j,s=0}^{\infty} A_{m,j,s} \cdot \frac{(f^*(x))^{\alpha+m+j+s}}{(2\lambda x)^{\alpha+m+j+s-1}}, \quad x > 0, \quad \alpha, \beta, \lambda > 0, \quad (14)$$

where $A_{m,j,s}$ is a constant term and it is defined as

$$A_{m,j,s} = (-1)^m \binom{\beta-1}{m} \binom{\alpha-1}{j} \frac{b_{s,j} \beta^\alpha}{\Gamma(\alpha)}.$$

Equation (14) gives the PDF of LGR distribution as a mixture of Rayleigh densities. So, several of its mathematical properties can be derived from those of the Rayleigh distribution.

3 Some Structural Properties

Established algebraic expansions to determine some mathematical properties of the LGR distribution can be more efficient than computing those directly by numerical integration (5), which can be prone to rounding off errors among others.

3.1 Quantile Function

The quantile function of is obtained by inverting (6) as

$$x = Q(u) = \left\{ -\frac{1}{\lambda} \log \left[1 - \exp \left(-\beta^{-1} \Gamma^{-1}(\alpha, u \Gamma(\alpha)) \right) \right] \right\}^{1/2},$$

where $\Gamma^{-1}(\cdot, \cdot)$ is the inverse incomplete gamma function implemented in most used mathematical software. Simulating the LGR random variable is straightforward. If U is a uniform variate on the unit interval $(0,1)$, then the random variable $X = Q(U)$ follows (6).

3.2 Moments

Using (12), the $r - th$ non-central moment of LGR can be obtained as

$$\mu'_r = E(X^r) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \Gamma\left(\frac{r+2}{2}\right) \cdot \sum_{m,j,s} (-1)^m \binom{\beta-1}{m} \binom{\alpha-1}{j} \frac{b_{s,j}}{\lambda^{*(r/2)}}. \quad (15)$$

3.3 Generating Function

The moment generating function, $M_X(t)$, of X can be obtained using (13) as follows:

$$M_X(t) = \int_0^\infty e^{tx} f(x) dx = \sum_{r=0}^\infty \frac{t^r}{r!} \cdot E(X^r),$$

hence, $M_X(t) = \sum_{m,j,s,r=0}^\infty B_{m,j,s,r} \cdot \frac{t^r}{r! \lambda^{*(r/2)}}$, where

$$B_{m,j,s,r} = (-1)^m \binom{\beta-1}{m} \binom{\alpha-1}{j} b_{s,j} \beta^\alpha \Gamma((r+2)/2) / \Gamma(\alpha).$$

4 Rényi and Shannon Entropies

The entropy represents a measure of uncertainty of a random variable. It is an important concept in many fields of science such as theory of communication, physics, engineering and economics, see Shannon [13] and Elshahhat et al. [14]. The expressions for the Shannon entropy and Rényi entropy will be provided of the LGR distribution.

4.1 Rényi Entropy

The Rényi entropy of the LGR distribution becomes

$$I_R(\theta) = (1-\theta)^{-1} \cdot \log \left\{ \sum_{t,m,s=0}^\infty Q_{t,m,s} \cdot \frac{\lambda^{-(\theta+1)/2} \Gamma((\theta+1)/2)}{2[\theta\alpha + m + t + s + 2]^{(\theta+1)/2}} \right\}, \quad \theta > 0, \quad \theta \neq 1.$$

for proof see Appendix (A).

4.2 Shannon Entropy

The Shannon's entropy of the LGR distribution will be

$$H_{Sh}(f(x)) = \log \Gamma(\alpha) + (1-\alpha)\psi(\alpha) - \log(\beta) + (\alpha-1)(\alpha/\beta) - \log(2\lambda) \\ + \sum_{i=0}^\infty (\beta/(i+1)(\beta+i+1))^\alpha - \frac{1}{2} \log \left\{ \frac{1}{\lambda} \cdot \sum_{i=0}^\infty (\beta/(i+1)(\beta+i+1))^\alpha \right\},$$

where $\psi(\alpha) = \frac{d(\log \Gamma(\alpha))}{d\alpha}$ is the digamma function, for proof see Appendix (B).

5 Order Statistics

Here, we provide the density of the $i - th$ order statistic $X_{(i)}$ in a random sample of n from the LGR distribution. The PDF of $X_{(i)}$ is defined as

$$f_n(x_{(i)}) = \frac{n!}{(i-1)!(n-i)!} [F(x_{(i)})]^{i-1} f(x_{(i)}) [1-F(x_{(i)})]^{n-i}, \quad -\infty < x_{(i)} < \infty. \quad (16)$$

Using (12) for $[1 - F(x_{(i)})]^{n-i}$, we can rewritten (16) in the following form

$$f_n(x_{(i)}) = Cf(x_{(i)}) \sum_{j=0}^{\infty} (-1)^j \binom{n-i}{j} [F(x_{(i)})]^{i+j-1}, \quad -\infty < x_{(i)} < \infty, \quad (17)$$

where, $C = \frac{n!}{((i-1)!(n-i)!)}$ is a constant. Substituting (6) in (17), we get

$$f_n(x_{(i)}) = Cf(x_{(i)}) \sum_{j=0}^{\infty} (-1)^j \binom{n-i}{j} \left[\frac{\Gamma(\alpha, -\beta \log(\eta(\lambda; x_{(i)})))}{\Gamma(\alpha)} \right]^{i+j-1}. \quad (18)$$

where, $\eta(\lambda; x_{(i)}) = 1 - e^{-\lambda x_{(i)}^2}$, $i = 1, 2, \dots, n$. Since,

$$\Gamma\{\alpha, -\beta \log(\eta(\lambda; x_{(i)}))\} = \Gamma(\alpha) - \gamma\{\alpha, -\beta \log(\eta(\lambda; x_{(i)}))\}, \quad (19)$$

where $\gamma\{\alpha, x\} = \int_0^x t^{\alpha-1} e^{-t} dt$, $x > 0$, $\alpha > 0$, is the lower incomplete gamma function.

Taking $\gamma(\alpha, x) = \sum_{r=0}^{\infty} (-1)^r x^{r+\alpha} / (r!(r+\alpha))$ for equation (19), we get

$$\Gamma\{\alpha, -\beta \log(\eta(\lambda; x_{(i)}))\} = \Gamma(\alpha) - \sum_{r=0}^{\infty} (-1)^r [-\beta \log(\eta(\lambda; x_{(i)}))]^{r+\alpha} / (r!(r+\alpha)). \quad (20)$$

From (20), equation (18) becomes

$$f_n(x_{(i)}) = Cf(x_{(i)}) \sum_{j=0}^{\infty} (-1)^j \binom{n-i}{j} \left[1 - \sum_{r=0}^{\infty} \frac{(-1)^r [-\beta \log(\eta(\lambda; x_{(i)}))]^{r+\alpha}}{r!(r+\alpha)\Gamma(\alpha)} \right]^{i+j-1}. \quad (21)$$

Using binomial expansion (10), then (21) can be rewritten as

$$\begin{aligned} f_n(x_{(i)}) &= Cf(x_{(i)}) \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{j+l} [\Gamma(\alpha)]^{-l} [-\beta \log(\eta(\lambda; x_{(i)}))]^{\alpha l} \\ &\quad \times \binom{n-i}{j} \binom{i+j-1}{l} \left[\sum_{r=0}^{\infty} \xi_r [-\beta \log(\eta(\lambda; x_{(i)}))]^r \right]^l, \end{aligned} \quad (22)$$

where $\xi_r = (-1)^r / r!(r+\alpha)$. Applying the power series raised to a positive integer, then (22) can be written as

$$\left[\sum_{r=0}^{\infty} \xi_r [-\beta \log(\eta(\lambda; x_{(i)}))]^r \right]^l = \sum_{r=0}^{\infty} d_{r,l} [-\beta \log(\eta(\lambda; x_{(i)}))]^r, \quad (23)$$

where $d_{0,l} = \xi_0^l$ and $d_{r,l} = (r\xi_0)^{-1} \sum_{w=1}^r [lw - r + w] \xi_w d_{r-w,l}$, see Pinho et al. [11]. Using (23), then (22) can be rewritten as follows

$$\begin{aligned} f_n(x_{(i)}) &= Cf(x_{(i)}) \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \binom{n-i}{j} \binom{i+j-1}{l} (-1)^{j+l} \\ &\quad \times [\Gamma(\alpha)]^{-l} [-\beta \log(\eta(\lambda; x_{(i)}))]^{\alpha l + r} d_{r,l}. \end{aligned} \quad (24)$$

Again, using (10), then the PDF of the i -th order statistic (24) becomes

$$f_n(x_{(i)}) = x_{(i)} \sum_{j,l,m,r=0}^{\infty} W_{j,l,m,r} (\Gamma(\alpha))^{-(l+1)} \left[-\log(\eta(\lambda; x_{(i)})) \right]^{\alpha(l+1)+r-1} \times e^{-(m+1)\lambda x_{(i)}^2} d_{r,l}, \quad 0 < x_{(i)} < \infty. \quad (25)$$

where W is a constant term and is given by

$$W_{j,l,m,r} = \frac{2n! \lambda \beta^{\alpha(l+1)+r}}{(n-i)!(i-1)!} \cdot \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \binom{n-i}{j} \binom{i+j-1}{l} \binom{\beta-1}{m} (-1)^{j+l+m}.$$

The PDF of the largest and smallest order statistics can be derived by replacing $(i = n)$ and $(i = 1)$ in (25), respectively, as

$$f_n(x_{(n)}) = 2n \lambda x_{(n)} \beta^{\alpha(l+1)+r} \cdot \sum_{l=0}^{n+j-1} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \binom{n+j-1}{l} \binom{\beta-1}{m} (-1)^{j+l+m} \times [\Gamma(\alpha)]^{-(l+1)} \left[-\log(\eta(\lambda; x_{(i)})) \right]^{\alpha(l+1)+r-1} e^{-(m+1)\lambda x_{(n)}^2} d_{r,l}, \quad 0 < x_{(n)} < \infty.$$

and

$$f_n(x_{(1)}) = 2n \lambda x_{(1)} \beta^{\alpha(l+1)+r} \cdot \sum_{j=0}^{n-1} \sum_{l=0}^j \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \binom{n-1}{j} \binom{j}{l} \binom{\beta-1}{m} (-1)^{j+l+m} \times [\Gamma(\alpha)]^{-(l+1)} \left[-\log(\eta(\lambda; x_{(i)})) \right]^{\alpha(l+1)+r-1} e^{-(m+1)\lambda x_{(1)}^2} d_{r,l}, \quad 0 < x_{(1)} < \infty.$$

6 Estimation and Fisher Information Matrix

In this section, we determine the maximum likelihood estimators (MLEs) of the unknown model parameters of LGR distribution based on complete samples. Let $x_1, x_2, \dots, x_n$ be the realizations of a random sample of size n from the LGR distribution. Then the log-likelihood function of the LGR distribution (5) for the vector of parameters $\Theta = (\alpha, \beta, \lambda)^T$ can be expressed as

$$\begin{aligned} \ell(\Theta) &= n \log(2\lambda\beta^\alpha) - n \log \Gamma(\alpha) - \lambda \sum_{i=1}^n x_i^2 + \sum_{i=1}^n \log(x_i) \\ &\quad + (\beta-1) \sum_{i=1}^n \log(\omega(x_i, \lambda)) + (\alpha-1) \sum_{i=1}^n \log(-\log(\omega(x_i, \lambda))). \end{aligned} \quad (26)$$

where, $\omega(x_i, \lambda) = 1 - e^{-\lambda x_i^2}$. Differentiating (26) partially with respect to α , β and λ , respectively, as

$$\begin{aligned} \frac{\partial \ell(\Theta)}{\partial \alpha} &= n \log(\beta) - n \psi(\alpha) + \sum_{i=1}^n \log(-\log(\omega(x_i, \lambda))), \\ \frac{\partial \ell(\Theta)}{\partial \beta} &= \frac{n\alpha}{\beta} + \sum_{i=1}^n \log(\omega(x_i, \lambda)), \\ \text{and} \\ \frac{\partial \ell(\Theta)}{\partial \lambda} &= \frac{n}{\lambda} - \sum_{i=1}^n x_i^2 + (\beta-1) \sum_{i=1}^n x_i^2 e^{-\lambda x_i^2} (\omega(x_i, \lambda))^{-1} \\ &\quad - (\alpha-1) \sum_{i=1}^n x_i^2 e^{-\lambda x_i^2} (\omega(x_i, \lambda))^{-1} \left(-\log \left[(\omega(x_i, \lambda))^{-1} \right] \right), \end{aligned} \quad (27)$$

where $\psi(\cdot)$ is the digamma function. Clearly, the maximum likelihood estimates of (27) can be obtained by solving set of non-linear equations, this needs computer facilities and numerical techniques. For interval estimation of the parameters, we require the 3×3 observed information matrix. Asymptotic variance-covariance matrix can be obtained by taking the inverse of the Fisher information matrix, which contains the second partial derivatives of the log-likelihood function (26) as follows:

$$\begin{aligned}\hat{\Sigma}(\hat{\Theta}) &= - \left[\begin{array}{ccc} \ell_{11} & & \\ \ell_{21} & \ell_{22} & \\ \ell_{31} & \ell_{32} & \ell_{33} \end{array} \right]_{(\alpha=\hat{\alpha}, \beta=\hat{\beta}, \lambda=\hat{\lambda})}^{-1} \\ &= \left[\begin{array}{ccc} \text{var}(\hat{\alpha}) & & \\ \text{cov}(\hat{\beta}, \hat{\alpha}) & \text{var}(\hat{\beta}) & \\ \text{cov}(\hat{\lambda}, \hat{\alpha}) & \text{cov}(\hat{\lambda}, \hat{\beta}) & \text{var}(\hat{\lambda}) \end{array} \right].\end{aligned}\quad (28)$$

The elements of the observed information matrix (28) will be

$$\begin{aligned}\ell_{33} &= -\frac{n}{\lambda^2} - (\beta - 1) \sum_{i=1}^n \left\{ x_i^4 \omega(x_i, \lambda) e^{-\lambda x_i^2} + x_i^4 \cdot e^{-2\lambda x_i^2} \right\} (\omega(x_i, \lambda))^{-2} \\ &\quad - (\alpha - 1) \sum_{i=1}^n \left\{ x_i^4 \cdot e^{-2\lambda x_i^2} \omega(x_i, \lambda)^{-2} \left[-\log(\omega(x_i, \lambda)) \right]^2 \right. \\ &\quad \left. - \left[-\log(\omega(x_i, \lambda)) \right]^{-1} \left\{ x_i^4 \cdot e^{-2\lambda x_i^2} (\omega(x_i, \lambda))^{-2} [1 + \omega(x_i, \lambda)] \right\} \right\}, \\ \ell_{11} &= -n\psi'(\alpha), \quad \ell_{22} = -\frac{n\alpha}{\beta^2}, \quad \ell_{12} = \frac{n}{\beta}, \quad \ell_{13} = -\sum_{i=1}^n x_i^2 \cdot e^{-\lambda x_i^2} (\omega(x_i, \lambda))^{-1} \left[-\log(\omega(x_i, \lambda)) \right]^{-1}, \\ \text{and} \\ \ell_{23} &= \sum_{i=1}^n x_i^2 e^{-\lambda x_i^2} (\omega(x_i, \lambda))^{-1}.\end{aligned}\quad (29)$$

The system of non-linear equations (29) can be solved numerically using statistical packages and the MLEs of α , β and λ , respectively, can be obtained. Under standard regularity conditions, the asymptotic distribution of the MLEs $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ is multivariate normal distribution and is given by

$$\hat{\Theta} \sim N \left[\Theta, \hat{\Sigma}(\hat{\Theta}) \right].$$

Solving the inverse matrix of (28) yields the asymptotic variance-covariance of the MLEs $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$. Thus, $100(1 - \gamma)\%$ two-sided approximate confidence intervals (CIs) for the unknown parameters α , β and λ can be obtained, respectively, as

$$\hat{\alpha} \pm z_{\gamma/2} \sqrt{\text{var}(\hat{\alpha})}, \quad \hat{\beta} \pm z_{\gamma/2} \sqrt{\text{var}(\hat{\beta})} \quad \text{and} \quad \hat{\lambda} \pm z_{\gamma/2} \sqrt{\text{var}(\hat{\lambda})},$$

where $z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $(\gamma/2) - th$.

7 Engineering Applications

In this section, we provide two applications to real data to illustrate the importance of the LGR model compared with other well-known lifetime models. Two real datasets are considered to fit the LGR distribution and compared with other known distributions, one based the breaking stress of carbon fibers data which presented by Nichols and Padgett [15]. The second real dataset was originally reported by

Leiblen and Zelen [16] which represents the number of revolutions to failure of ball bearings. We compare the fit of the LGR model with those of other competitive models, namely: Rayleigh distribution (RD), log-Rayleigh distribution (LRD), see Rivet et al. [17], log-gamma distribution (LGD), gamma-Rayleigh distribution (GRD), see Provost et al. [18].

In order to compare the fitted models, we consider some measures of goodness-of-fit including: the log-likelihood function ($-\ell$) evaluated at the MLEs, Akaike information criterion (AIC), $AIC = -2\ell + 2p$, corrected Akaike information criterion (AICC), $AICC = AIC + \frac{2np}{n-p-1}$, consistent Akaike information criterion (CAIC), $CAIC = -2\ell + \frac{2np}{n-p-1}$, Bayesian information criteria (BIC) $BIC = -2\ell + p \log(n)$, and Hannan-Quinn information criterion (HQIC), $HQIC = -2\ell + 2p \log(\log(n))$, where n and p are the sample size and the number of model parameters, respectively. Further, we adopt the Anderson-Darling (A^*), Cramér-von Mises (W^*) and Kolmogorov-Smirnov (K-S) test statistics are computed to compare the fitted models. The statistics A^* and W^* are described in details in Chen and Balakrishnan [19]. In general, the smaller values of these statistics, the better the fit to the data. The required programming figures are carried out in the R-language introduced by R Development Core Team [20] and Nadarajah and Rocha [21]. The R script used to fit the LGR parameters is provided in Appendix (C).

Application 1: Nichols and Padgett Real Dataset

The observations on breaking stress of carbon fibers (in Gba) with $n = 100$ are listed in Table 1 and the summary of this data was reported in Table 2.

Table 1: Dataset of breaking stress of carbon fibers

0.39	0.81	0.85	0.98	1.08	1.12	1.17	1.18	1.22	1.25
1.36	1.41	1.47	1.57	1.57	1.59	1.59	1.61	1.61	1.69
1.69	1.71	1.73	1.80	1.84	1.84	1.87	1.89	1.92	2.00
2.03	2.03	2.05	2.12	2.17	2.17	2.17	2.35	2.38	2.41
2.43	2.48	2.48	2.50	2.53	2.55	2.55	2.56	2.59	2.67
2.73	2.74	2.76	2.77	2.79	2.81	2.81	2.82	2.83	2.85
2.87	2.88	2.93	2.95	2.96	2.97	2.97	3.09	3.11	3.11
3.15	3.15	3.19	3.19	3.22	3.22	3.27	3.28	3.31	3.31
3.33	3.39	3.39	3.51	3.56	3.60	3.65	3.68	3.68	3.68
3.70	3.75	4.20	4.38	4.42	4.70	4.90	4.91	5.08	5.56

Table 2: Summary for breaking stress of carbon fibers data

n	Min.	1.st Qu.	Median	Mean	3.rd Qu.	Max.
100	0.39	1.84	2.7	2.621	3.22	5.56

Using the dataset of Table 1, we compare the LGR distribution with Rayleigh and exponential lifetime models. Applying the K-S one sample test for the real data in Table 1, we get $D = 0.117$ is lower than the K-S tabulated value 0.136, on the other hand, the associated p-value is 0.127 is quite high of the significance level 0.05, then we cannot reject the null hypothesis that the data are follows the LGR distribution. Table 3 lists the MLEs and their corresponding standard errors (SEs, in parentheses) of the LGR, Rayleigh and exponential parameters. The numerical values of statistics -2ℓ , AIC, CAIC, BIC and HQIC are listed in Table 4. Tables 3 and 4 compare the LGR model with the Rayleigh and exponential models. We note that the LGR model gives the lowest values for all model selection statistics among all fitted models. So, the LGR model is the best model for breaking stress of carbon fibers data. The histogram of the breaking stress of carbon fibers dataset and their estimated PDFs and CDFs for the fitted models is displayed in Figures 3

and 4, respectively. These plots indicate that the LGR distribution gives a better fit to the histograms and therefore could be chosen to explain this dataset.

Table 3: The fitting MLEs and 95% ACIs from carbon fiber dataset.

Model	MLE					
	$\hat{\lambda}$		$\hat{\alpha}$		$\hat{\beta}$	
	Est. (SE)	95% ACI	Est. (SE)	95% ACI	Est. (SE)	95% ACI
LGR	1.1201 (0.0060)	(1.1079,1.1322) (2.1081,3.1350)	0.1319 (0.0141)	(0.1042,0.1593) -	2.0944 (0.6182)	(0.8842,3.3051) -
Exp	2.6214 (0.2621)	(0.1023,0.1521) -	- -	- -	- -	- -
Ray	0.1268 (0.0132)	- -	- -	- -	- -	- -

Table 4: Fitting summary of LGR and its competitors from carbon fibers data

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC
LGR	287.56	293.57	293.82	301.38	296.73
Exp	392.74	394.74	394.78	401.95	395.79
Ray	299.01	301.01	301.04	308.21	302.06

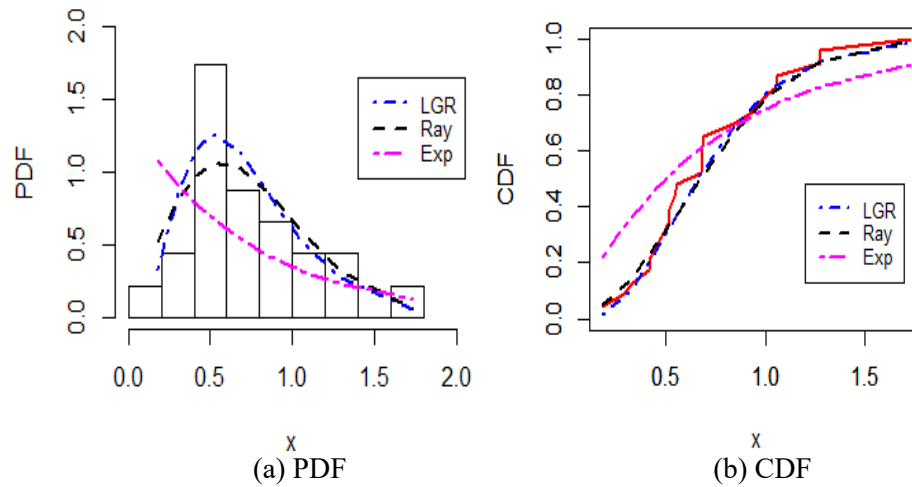


Figure 3: The fitted PDF/CDF lines for LGR and its competitors from carbon fibers data.

Application 2: Ball Bearings Real Dataset

Leiblen and Zelen [16] performed life tests and determined the number of revolutions to failure for 23 ball bearings. Recently, this dataset was reanalyzed by Elshahhat et al. [22], Alotaibi et al. [23], and Haj Ahmad and Elshahhat [24]. The ordered data with $n = 23$ observations are listed in Table 5 and the summary of this data was reported in Table 6. Using the dataset of Table 5, we compare the LGR distribution with Rayleigh and exponential lifetime models. One question arises about whether the dataset follows the LGR distribution or not. To check for the goodness of fit, we consider the K-S test statistic. Applying the K-S one sample test for the real data in Table 5, we get $D = 0.119$ is lower than the K-S tabulated value 0.136, on the other hand, the associated p-value is 0.897 is quite high of the significance level 0.05, then we cannot reject the null hypothesis that the ball bearings data are follow the LGR distribution.

Table 5: Dataset of failure for ball bearings

0.1788	0.2852	0.3300	0.4152	0.4212	0.4560	0.4848	0.5186
0.5196	0.5412	0.5556	0.6780	0.6864	0.6864	0.6888	0.8412
0.9312	0.9864	1.0512	1.0584	1.2792	1.2804	1.7340	

Table 6: Summary for failure for ball bearings data

n	Min.	1.st Qu.	Median	Mean	3.rd Qu.	Max.
23	0.1788	0.4704	0.678	0.7221	0.9588	1.734

Measures of goodness of fit including: -2ℓ , AIC, CAIC, BIC and HQIC are computed to compare the fitted models. In general, the smaller the values of these statistics, the better the fit to the data. Table 7 lists the MLEs and their corresponding standard errors (in parentheses) of the LGR, Rayleigh and exponential parameters. The numerical values of statistics -2ℓ , AIC, CAIC, BIC and HQIC are listed in Table 8. Tables 3 and 4 compare the LGR model with the Rayleigh and exponential models. We note that the LGR model gives the lowest values for all fitting statistics among all fitted models. So, the LGR model could be chosen as the best model for number of revolutions to failure for ball bearings dataset.

Table 7: The fitting MLEs and 95% ACIs from ball bearings dataset.

Model	MLE					
	$\hat{\lambda}$		$\hat{\alpha}$		$\hat{\beta}$	
	Est. (SE)	95% ACI	Est. (SE)	95% ACI	Est. (SE)	95% ACI
LGR	0.0281 (0.074)	(0.000, 0.174) (0.819, 1.951)	18.303 (21.771)	(0.000, 60.972)	4.0776 (2.6191)	(0.000, 9.210)
Exp	1.3849 (0.289)	(0.902, 2.147) -	- -		- -	
Ray	1.5244 (0.318)	-	- -		- -	

Table 8: Fitting summary of LGR and its competitors from ball bearings data

Distribution	-2ℓ	AIC	CAIC	BIC	HQIC
LGR	14.155	20.155	21.418	23.562	21.012
Exp	31.021	33.021	33.212	37.292	33.021
Ray	28.248	30.248	30.439	34.519	30.534

The histogram of the number of failure for ball bearings dataset and their estimated PDFs and CDFs for the fitted models are displayed in Figures 5 and 6, respectively. These plots indicate that the LGR distribution gives a better fit to the histograms and therefore could be chosen to explain these datasets.

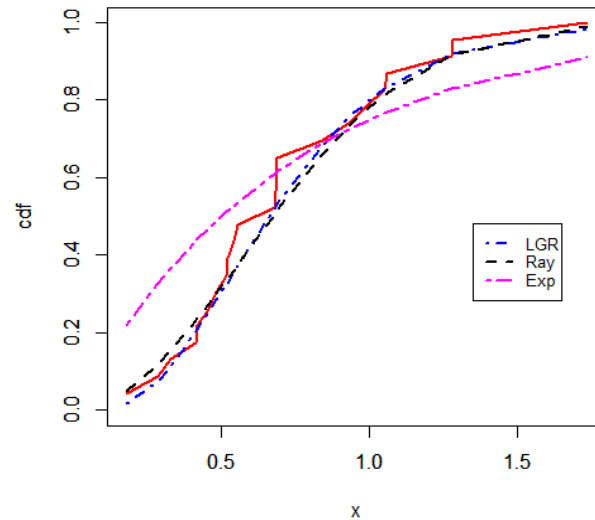


Figure 4: The fitted CDF lines for LGR and its competitors from ball bearings data.

8 Conclusion

In this study, we introduced a new three-parameter extension of the Rayleigh distribution, termed the LGR model, and examined its theoretical, inferential, and applied properties. By integrating the log-gamma generator into the Rayleigh baseline, the proposed model substantially enhances flexibility, enabling it to accommodate a wide spectrum of distributional forms and hazard rate behaviors commonly observed in lifetime and reliability data. A key theoretical contribution is the derivation of a series expansion showing that the LGR density can be expressed as a mixture of Rayleigh densities. This representation simplifies the derivation of many mathematical properties and removes dependence on heavy numerical methods. Using this expansion, explicit expressions were obtained for the quantile function, non-central moments, and moment generating function, along with closed-form formulas for Rényi and Shannon entropies. From an inferential perspective, maximum likelihood estimation and the observed Fisher information were derived, enabling construction of asymptotic variance–covariance matrices and approximate confidence intervals for model parameters. These results support efficient numerical implementation and ensure statistical reliability for parameter estimation. Two real engineering datasets, carbon-fiber breaking stresses and ball-bearing failures, were analyzed to demonstrate the practical relevance of the LGR model. In both cases, the LGR model outperformed competing models, including the Rayleigh and exponential distributions, across multiple criteria. Overall, the findings confirm that the LGR distribution is a flexible, robust, and analytically tractable model with strong potential for applications in reliability, survival analysis, and other fields.

Funding Statement

The author declared that no specific funding was received for this work.

Data Availability

The authors confirm that the data supporting the findings of this study are available within the article.

Conflicts of Interest:

The authors declare no conflicts of interest regarding this study.

AI Usage Statement:

The authors declare that they have not used AI tools in the creation of this article.

Appendix

Appendix A: Rényi entropy

The entropy of a random variable X with PDF, $f(x)$, is a measure of variation of the uncertainty, see Rényi [25]. The Rényi entropy is given by

$$I_R(\theta) = \frac{1}{1-\theta} \log \left[\int_{-\infty}^{\infty} f(x)^\theta dx \right], \quad \theta > 0, \quad \theta \neq 1.$$

Using the same steps of the LGR expansion series, one write $f(x)^\theta$ as

$$I_R(\theta) = \frac{1}{1-\theta} \cdot \log \left[\frac{(2\lambda\beta^\alpha)^\theta}{(\Gamma(\alpha))^\theta} \int_0^\infty x^\theta e^{-\theta\lambda x^2} (1-e^{-\lambda x^2})^{\theta(\beta-1)} \left[-\log(1-e^{-\lambda x^2}) \right]^{\theta(\alpha-1)} dx \right],$$

then, we have

$$I_R(\theta) = \frac{1}{1-\theta} \cdot \log \left[\sum_{t,m,s=0}^{\infty} Q_{t,m,s} \cdot \int_0^\infty x^\theta \exp(-\lambda x^2 (\theta\alpha + m + t + s + 2)) dx \right], \quad \theta > 0, \quad \theta \neq 1.$$

where Q is a constant term and is given by

$$Q_{t,m,s} = \frac{(2\lambda\beta^\alpha)^\theta}{(\Gamma(p))^\theta} \sum_{t=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} (-1)^t \binom{\theta(\beta-1)}{t} \binom{\theta(\alpha-1)}{m} b_{s,m}.$$

Let $W = \lambda x^2 [\theta\alpha + m + t + s + 2]$, $0 < w < \infty$.

Then, the Rényi entropy of the LGR distribution becomes

$$I_R(\theta) = \frac{1}{1-\theta} \cdot \log \left[\sum_{t,m,s=0}^{\infty} Q_{t,m,s} \cdot \frac{\lambda^{-(\theta+1)/2} \cdot \int_0^\infty w^{(\theta-1)/2} e^{-w} dw}{2 [\theta\alpha + m + t + s + 2]^{(\theta+1)/2}} \right], \quad \theta > 0, \quad \theta \neq 1.$$

Appendix B: Shannon entropy

Shannon [13] introduced the most popular measure of entropy which is called Shannon entropy. For a continuous distribution $F(x)$ with density $f(x)$, the Shannon entropy is defined as

$$H_{Sh}(f(x)) = E[-\log(f(x))].$$

Lemma: The Shannon's entropy for a random variable X that follows the LG-G family (2) is given by

$$H_{Sh}(f(x)) = \log \Gamma(\alpha) + (1-\alpha)\psi(\alpha) - \log(\beta) + (\beta-1)(\alpha/\beta) - E_Y \left\{ \log \left(g \left[G^{-1}(e^{-y}) \right] \right) \right\},$$

where $Y \sim \text{Gamma}(\alpha, \beta)$ and $\psi(\cdot)$ denotes the digamma function, see Ashour et al. (2014). From (3) the inverse Rayleigh distribution is

$$G^{-1}(x) = \left(-\frac{1}{\lambda} \cdot \log(1-x) \right)^{1/2}. \quad (\text{B.1})$$

Using equations (B.1) and (4), we get

$$g(G^{-1}(e^{-y})) = g \left(\left[-\frac{1}{\lambda} \cdot \log(1-e^{-y}) \right]^{1/2} \right) = 2\lambda(1-e^{-y}) \left[-\frac{1}{\lambda} \cdot \log(1-e^{-y}) \right]^{1/2}.$$

Taking the logarithm for both two sided, we get

$$\log \left[g \left(G^{-1} \left(e^{-y} \right) \right) \right] = \log(2\lambda) + \log(1 - e^{-y}) + \frac{1}{2} \log \left[-\frac{1}{\lambda} \cdot \log(1 - e^{-y}) \right]. \quad (\text{B.2})$$

Following Pinho et al. [11], the following power series expansion will be used

$$-\log(1 - e^{-y}) = \sum_{i=0}^{\infty} \frac{e^{-y(i+1)}}{i+1}. \quad (\text{B.3})$$

Using (B.2) and taking the expectation for two sided of (B.3), we get

$$\begin{aligned} E_Y \left(\log \left[g \left(G^{-1} \left(e^{-y} \right) \right) \right] \right) &= \log(2\lambda) - \sum_{i=0}^{\infty} \frac{1}{i+1} \cdot E_Y \left(e^{-y(i+1)} \right) \\ &+ \frac{1}{2} \log \left[\frac{1}{\lambda} \cdot \sum_{i=0}^{\infty} \frac{1}{i+1} \cdot E_Y \left(e^{-y(i+1)} \right) \right]. \end{aligned} \quad (\text{B.4})$$

If $Y \sim \text{Gamma}(\alpha, \beta)$, then

$$E_Y \left(e^{-y(i+1)} \right) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \int_0^\infty y^{\alpha-1} e^{-y(\beta+i+1)} dy = \left(\beta / (\beta + i + 1) \right)^\alpha. \quad (\text{B.5})$$

Substituting (B.5) in (B.4), we find that

$$\begin{aligned} E_Y \left[\log \left(g \left(G^{-1} \left(e^{-y} \right) \right) \right) \right] &= \log(2\lambda) - \sum_{i=0}^{\infty} \left(\beta / [(i+1)(\beta+i+1)] \right)^\alpha \\ &+ \frac{1}{2} \log \left[\frac{1}{\lambda} \cdot \sum_{i=0}^{\infty} \left(\beta / [(i+1)(\beta+i+1)] \right)^\alpha \right]. \end{aligned} \quad (\text{B.6})$$

Finally, substituting (B.6) in the equation of the Shannon's entropy form, the Shannon's entropy of the LGR distribution will be

$$\begin{aligned} H_{Sh} \left(f(x) \right) &= \log \Gamma(\alpha) + (1 - \alpha) \psi(\alpha) - \log(\beta) + (\alpha - 1)(\alpha/\beta) - \log(2\lambda) \\ &+ \sum_{i=0}^{\infty} \left(\beta / (i+1)(\beta+i+1) \right)^\alpha - \frac{1}{2} \log \left\{ \frac{1}{\lambda} \cdot \sum_{i=0}^{\infty} \left(\beta / (i+1)(\beta+i+1) \right)^\alpha \right\}, \end{aligned}$$

where $\psi(\alpha) = \frac{d(\log \Gamma(\alpha))}{d\alpha}$ is the digamma function.

Appendix C: The R codes

The following R script used to fit the LGR parameters, to calculate the model selection criteria, and to calculate the K-S (p-value) distance and the proposed fitting criteria.

```
pdf_LGR <- function(par,x) {
  alpha = par [1]
  beta  = par [2]
  lambda = par [3]
  term1 <- 2 * lambda * beta^alpha * x * exp(-lambda * x^2) / gamma(alpha)
  term2 <- (1 - exp(-lambda * x^2))^(beta - 1)
  term3 <- (-log(1 - exp(-lambda * x^2)))^(alpha - 1)
  result <- term1 * term2 * term3
  return(result)
}
cdf_LGR <- function(par,x) {
  alpha = par [1]
  beta  = par [2]
  lambda = par [3]
  term  = -beta * log(1 - exp(-lambda * x^2))
```

```

result <- 1 - pgamma(term, shape = alpha, lower.tail = TRUE)
gamma / Gamma(alpha)
return(result)
}
LGR <- goodness.fit(PDF = pdf_LGR, CDF = cdf_LGR, start=c(0.1,0.1,0.1), data,
method = "BFGS", domain = c (0,Inf), mle = NULL)

print (LGR)

```

References

- [1] Amini, M., MirMostafaei, S. M. T. K., and Ahmadi, J. (2014). Log-gamma-generated families of distributions. *Statistics*. 48(4), 913-932.
- [2] Ferreira, J. T., Bekker, A., and Arashi, M. (2016). Objective Bayesian estimators for the right censored Rayleigh distribution: evaluating the Al-Bayyati loss function. *REVSTAT-Statistical Journal*. 14(4), 433-454.
- [3] Gupta, I. (2017). Bayesian and E-Bayesian method of estimation of parameter of Rayleigh distribution-a Bayesian approach under linex loss function. *International Journal of Statistics and Systems*. 12(4), 791-796.
- [4] Kotb, M. S., and Raqab, M. Z. (2018). Bayesian inference and prediction of the Rayleigh distribution based on ordered ranked set sampling. *Communications in Statistics-Simulation and Computation*. 47(3), 905-923.
- [5] Dey, S., Salehi, M., and Ahmadi, J. (2017). Rayleigh distribution revisited via ranked set sampling. *Metron*. 75(1), 69-85.
- [6] Ashour, S. K., Said, D. R. and Fahim, M. A. A., (2014). The log-gamma-Pareto distribution. *International Journal of Sciences: Basic and Applied Research*. 16, 357-374.
- [7] Cordeiro, G. M., Maria do Carmo, S. L., Gomes, A. E., da-Silva, C. Q. & Ortega, E. M. (2016). The gamma-extended Weibull distribution. *Journal of Statistical Distributions and Applications*. 3:7. DOI 10.1186/s40488-016-0043- .
- [8] Cordeiro, G. M., Ortega, E. M., and Popović, B. V. (2015). The gamma-Lomax distribution. *Journal of Statistical computation and Simulation*. 85(2), 305-319.
- [9] Castellares, F., Santos, M. A., Montenegro, L. C., and Cordeiro, G. M. (2015). A gamma-generated logistic distribution: properties and inference. *American Journal of Mathematical and Management Sciences*. 34, 14-39.
- [10] Dias Gonçalves, J. H., Ferreira Gomes, J. J., Rubio, L., and Ramos, F. R. (2023). A generalized log gamma approach: Theoretical contributions and an application to companies' life expectancy. *Mathematics*. 11(23), 4792.
- [11] Pinho, L. G. B., Cordeiro, G. M. and Nobre, J. S. (2012). The gamma-exponentiated Weibull distribution. *Journal of Statistical Theory and Applications*. 11, 379-395.
- [12] Gradshteyn, I. S., and Ryzhik, I. M. (2007). *Table of Integrals, Series, and Products*. Academic, New York, 1965.
- [13] Shannon, C. E. (1948). A mathematical theory of communication, *Bell System Technical Journal*. 27, 379-423.
- [14] Elshahhat, A., El-Sherpieny, E. S. A., and Hassan, A. S. (2023). The Pareto–Poisson distribution: characteristics, estimations and engineering applications. *Sankhya A*. 85(1), 1058-1099.
- [15] Nichols, M. D., and Padgett, W. J. (2006). A bootstrap control chart for Weibull percentiles. *Quality and Reliability Engineering International*. 22, 141–151.
- [16] Lieblein, J. and Zelen, M. (1956). Statistical investigation of the fatigue life of deep-groove ball bearings. *Journal of Research of the National Bureau of Standards*. 57, 273-316.
- [17] Rivet, B., Girin, L., and Jutten, C. (2007). Log-Rayleigh distribution: a simple and efficient statistical representation of log-spectral coefficients. *IEEE transactions on audio, speech, and language processing*. 15(3), 796-802.

- [18] Provost, S. B., Saboor, A., and Ahmad, M. (2011). The gamma-Weibull distribution. *Pakistan Journal of Statistics*. 27(2), 111-131.
- [19] Chen, G., and Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*. 27(2), 154-161.
- [20] R Development Core Team (2009). "R: A Language and Environment for Statistical Computing, R Foundation for Statistical Computing." Vienna, Austria.
- [21] Nadarajah, S., and Rocha, R. (2016). Newdistns: An R package for new families of distributions. *Journal of Statistical Software*. 69, 1-32.
- [22] Elshahhat, A., Bhattacharya, R., and Mohammed, H. S. (2022). Survival analysis of Type-II Lehmann Fréchet parameters via progressive type-II censoring with applications. *Axioms*. 11(12), 700.
- [23] Alotaibi, R., Rezk, H., and Elshahhat, A. (2024). Evaluations of Chris-Jerry data using generalized progressive hybrid strategy and its engineering applications. *CMES-Computer Modeling in Engineering and Sciences*. 140(3), 3073-3103.
- [24] Haj Ahmad, H., and Elshahhat, A. (2025). A new Hjorth distribution in its discrete version. *Mathematics*. 13(5), 875.
- [25] Rényi, A. (1961). On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, 1, Contributions to the Theory of Statistics. The Regents of the University of California.

Disclaimer / Publisher's Note:

The statements and data in all publications are solely those of the author(s) and contributor(s) and do not necessarily reflect the views of the publisher. The publisher accepts no responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.